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# Measures of Income Segregation

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**Abstract**

In this paper I propose a class of measures of rank-order segregation, each of which may be used to measure segregation by a continuous (but not necessarily interval-scaled) variable, such as income. These rank-order segregation indices have several appealing features that remedy flaws in existing measures of income segregation. First, the measures are insensitive to rank-preserving changes in the income distribution. As a result, the measures are independent of the extent of income inequality and allow comparisons across place and time regardless of the units of income or differences in the cost of living. Second, the measures can be easily computed from either exact or categorical income data, and are largely insensitive to variation in how income is tabulated in Census data. Third, the measures satisfy a number of mathematical properties necessary or desirable in such indices. Fourth, the measures are easily adapted to account for spatial proximity. Finally, the indices can be interpreted in a variety of equivalent ways that illustrate their correspondence with standard notions of segregation. I illustrate the computation and interpretation of these measures using Census data from two U.S. cities: San Francisco, CA and Detroit, MI.

## Introduction

In any city in the world, even a cursory inspection of residential patterns—by any non-sociologist, non-economist, or non-geographer—would indicate the presence of some degree of residential segregation by income and wealth. There are some neighborhoods populated primarily by families with above-average income and wealth, and other neighborhoods populated primarily by families with below average income and wealth. For the sociologist, economist, or geographer, however, it is not enough to merely note the presence of such residential sorting; we desire as well to quantify it. Virtually any interesting question regarding the causes, patterns, and consequences of such residential segregation requires that we measure it—and measure it in a way that makes comparisons across places and times meaningful.

Surprisingly, the set of tools available to scholars for measuring spatial economic segregation is relatively limited. Our goal in this paper is to develop an approach to measuring economic segregation that is intuitively meaningful, easy to compute, and allows for comparisons across place and time. I draw intuition from the methods of measuring segregation along an ordinal dimension developed in Reardon (2009), and extend these methods to apply to segregation along a continuous dimension. Although I initially develop 'aspatial' versions of these measures, I show how they can be easily adapted to take into account the spatial or social proximity of individuals by using the approach outlined by Reardon and O'Sullivan (2004).

I begin by reviewing and critiquing existing methods of measuring income and economic segregation. Following this review, I outline a set of desirable properties of measures of income segregation (or, more generally, of segregation along a continuous variable).[1] To provide intuition for the development of income segregation measures, I begin by describing the class of measures of ordinal segregation developed by Reardon (2009). Building on this intuition, I develop a related

[1] Although the measures I discuss in this paper can be used to characterize segregation along any continuous variable, throughout the paper I refer to them as measures of "economic segregation" or "income segregation" in order to make our discussion more concrete.

class of measures—measures of rank-order segregation—that can be used when the variable of interest is continuous, rather than ordinal. In this section I also describe a computationally simple method of computing these measures even when income data are reported categorically rather than continuously. Finally, I demonstrate that the indices satisfy the criteria proposed in the second section of the paper. Throughout the paper, I illustrate the intuition, computation, and interpretation of the measures using data from two U.S. cities that have very different income distributions: San Francisco, CA and Detroit, MI.

## 1. Existing Measures of Economic Segregation

Prior research on economic segregation has relied on several general approaches for characterizing the extent to which individuals of different socioeconomic characteristics are unevenly distributed throughout a region. Most of this research has been concerned with income segregation, rather than segregation by wealth, largely because income data are far more readily available. This research comes primarily from three different disciplinary perspectives—sociology, economics, and geography—each of which faces the same set of measurement issues.

In most data sources, income data are reported categorically, as counts within each organizational unit (e.g., census tract) of households, families, or individuals falling in a set of mutually exclusive and exhaustive ordered income categories. Each of these income categories is defined by a pair of upper and lower income bounds (except for the two extreme categories, which are each unbounded on one side).[2] As a result, the measurement of income segregation is hampered by the fact that we lack full information on the income distribution overall or in any one organizational unit. Moreover, although the income thresholds that define the ordered income

[2] For example, in the 2000 U.S. census, annual household income is categorized by 16 income ranges, ranging from "less than $10,000," "$10,000-$14,999," "$15,000-$19,000," etc., through "$150,000-$199,999," and finally "$200,000 or more." Because some households have negative income, the lowest income category is technically not bounded by 0 (although the Census treats households with negative income as having zero income in some cases).

categories are designed to span most of the range of incomes, they are nonetheless relatively arbitrary and change over time. Any segregation measure that relies on them should be insensitive to the definition of the thresholds if the measure is to be useful for comparative purposes.

*Category-Based Measures of Income Segregation*

By far the most common method of measuring income segregation used in existing research has been to divide the population into two categories based on some chosen income threshold (a wide range of thresholds are used in extant research). Segregation between these two groups (those above and those below the chosen threshold) is computed using any conventional two-group segregation measure, such as the dissimilarity index. Examples of this approach are found in the literature in sociology (Fong & Shibuya, 2000; Massey, 1996; Massey & Eggers, 1993; Massey & Fischer, 2003), urban planning (Coulton, Chow, Wang, & Su, 1996; Pendall & Carruthers, 2003), and economics (Jenkins, Micklewright, & Schnepf, 2006; Waitzman & Smith, 1998).

Although the primary advantage of this approach is its simplicity, its shortcomings are several and obvious. First, dichotomizing the income distribution discards a substantial amount of information. Even if we do not know households' exact incomes, the 16 income categories reported in the 2000 U.S. census, for example, contain far more information than any dichotomized version. Second, the results of such an approach may depend on the choice of a threshold—segregation between the very poor and everyone else may not be (and generally is not) the same as segregation between the very rich and everyone else.

Several variants of this approach have been used. Massey and Fischer (2003), for example, compute segregation between poor and affluent households (ignoring the middle-class) to better capture the separation of the extremes of the income distribution. A second variant of the categorical approach to measuring income segregation is to compute the two-group segregation for many or all possible pairs of income categories, and then to construct some average or summary measure of these multiple pairwise indices (Farley, 1977; Massey & Eggers, 1990; Telles, 1995). Third, rather than dichotomize the income distribution, Fong and Shibuya (2000) and Telles (1995) compute segregation among multiple income category groups using the Theil information theory index of segregation (Theil, 1972). This approach, however, uses an index designed to measure segregation among a set of *unordered* groups (such as race/ethnic groups) to measure segregation among a set of *ordered* groups (income groups), and so is insensitive to the inherently ordered nature of income categories. Finally, Meng, Hall, and Roberts (2006) measure the segregation among multiple ordered income groups using an approach that explicitly accounts for the ordered nature of the categories by weighting the segregation between different groups by some measure of the 'social distance' between the groups. While many of these variants have some advantage over simply dichotomizing the income distribution, each is sensitive to the number and location of the thresholds used to define income categories, confounding the possibility of making meaningful comparisons across places and times.

*Variation-Ratio Measures of Income Segregation*

A second approach to measuring economic segregation defines segregation as a ratio of the between-neighborhood variation in mean income or wealth to the total population variation in income or wealth. Some measures derived from this approach use the variance of incomes as the measure of income variation (Davidoff, 2005; Wheeler, 2006; Wheeler & La Jeunesse, 2006), and so have an interpretation analogous to the $R^2$ statistic from a regression of individual incomes on a set of neighborhood dummy variables. Similarly, Jargowsky (1996, 1997) defines income segregation as the ratio of the between-unit (e.g., between-tract) standard deviation of income to the overall regional income standard deviation. Others use a measure of income inequality as a measure of variation. Kim & Jargowsky (2009), for example, use the Gini index as a measure of variation; Ioannides (2004) uses the ratio of the variance of log incomes; Hardman & Ioannides (2004) use the ratio of within-neighborhood to overall coefficients of variation of income; and Ioannides and Seslen (2002) use the ratio of Bourguignon's population-weighted decomposable inequality index to measure both income and wealth segregation. The relative merits and flaws of the choice of inequality or variation measure used to construct the various ratio-based indices have not been fully investigated.

In principle, measures based on this approach use full information on the income distribution in each census tract or organizational unit, but because exact income distribution data are generally not available, they must rely in part on the estimation of parameters describing the overall income distribution (see, e.g., Jargowsky, 1996; Wheeler & La Jeunesse, 2006). This estimation, in turn, may be very sensitive to assumptions about the income levels of individuals in the top income category. Moreover, it is not clear how sensitive the measurement of income segregation is to the choice of a measure of income spread (variance, standard deviation, variance of log income, coefficient of variation, Bourguignon inequality, etc.); nor is it clear on what basis one should choose among these. The standard deviation or variance ratio measures (as used, for example, by Jargowsky, 1996, 1997; Wheeler, 2006; Wheeler & La Jeunesse, 2006) are insensitive to shape-preserving changes in the income distribution (changes that add and/or multiply all incomes by a constant), but measures based on other parameters, such as the variance of logged incomes are not (e.g., the variance of logged income is insensitive to constant multiplicative changes in incomes, but is sensitive to changes that add a constant to all incomes).

To the extent that the required parameters (e.g., variance) of the income distribution can be estimated well from the reported counts by income category, variation ratio approaches have considerably more appeal than existing approaches that rely on computing pairwise segregation between groups defined by one or more income thresholds. They use (in theory) complete information on the income distribution; they do not rely on arbitrary threshold choices; and at least some such measures are invariant to certain types of changes in the income distribution.

One interesting version of the variation ratio approach is developed by Watson (2009), who measures income segregation using the Centile Gap Index ($CGI$), which is defined as one minus the ratio of a measure of the within-neighborhood variation in income *percentile ranks* to the overall variation in income percentile ranks. Specifically, the $CGI$ measures within-neighborhood variation in percentile ranks as the mean absolute deviation of households' income percentile from the percentile rank of their neighborhood median. The $CGI$, because it is based on variation in percentile ranks rather than income levels, is insensitive to any rank-preserving changes in the income distribution, a desirable feature that other ratio measures lack.

Nonetheless, while Watson's $CGI$ has some appeal because of its insensitivity to rank-preserving changes in income, it has a subtle flaw. It is insensitive to some types of redistributions of individuals among neighborhoods that should intuitively increase segregation. For example, if we have a region consisting of two neighborhoods with identical income distributions (so that $CGI = 0$), and we rearrange households so that one neighborhood consists of the households in the first and third quartiles of each prior neighborhood and the other consists of the households in the second and fourth quartiles of each prior neighborhood, the $CGI$ will be unchanged ($CGI = 0$), despite the fact that we have created an uneven distribution of households among neighborhoods, such that the two neighborhoods now have different income distributions.

*Spatial Autocorrelation Measures of Income Segregation*

In general, most proposed measures of income segregation are *aspatial*—that is, they do not account for the spatial proximity of individuals/households, except insofar as spatial proximity is accounted for by census or administrative area boundaries. A third approach to measuring income segregation derives from the geographical notion of spatial autocorrelation (see Anselin, 1995; Cliff & Ord, 1981). In this approach, which explicitly accounts for the spatial patterning of households, segregation is conceived as the extent to which households near one another have more similar incomes than those that are farther from one another. Although several measures that account for the spatial proximity of census tracts, for example, have been suggested (Chakravorty, 1996; Dawkins, 2007; Jargowsky & Kim, 2004), this approach to measuring income segregation is the least well-developed. Moreover, although these measures have the advantage of being explicitly spatial, they are subject to the modifiable areal unit problem (MAUP) (Openshaw, 1984), meaning that they are sensitive to areal unit boundary definitions because they rely on a definition of spatial proximity defined by the somewhat arbitrary size and shapes of census tracts. Even in the absence of MAUP issues, the measurement properties of spatial autocorrelation indices of income segregation are not well understood.

In sum, while a wide range of measures have been used to describe income segregation, several key flaws plague existing measures. Measures based on computing categorical segregation indices among income categories, while widely used because of their ease of computation, are inherently sensitive to changes in both the choice of thresholds (as occurs between censuses in the U.S.) and differences in income distributions (either regional or temporal), even in the absence of any change in the location and relative income levels of households. Measures based on ratios of income variation within and among locations, in contrast, do not depend on the definition of income categories, at least in principle. In practice, however, the distributional parameters used in such measures must be estimated from the categorical income data generally reported, and so may be sensitive not only to the definition of categories, but also to assumptions about the shape of the income distribution, particularly for the highest-earning category, which has no upper bound. Finally, indices based on spatial autocorrelation or spatial proximity generally rely on relatively *ad hoc* definitions of proximity and have measurement properties that are not well understood.

## 2. Desirable Properties of an Income Segregation Measure

Our aim in this paper is to develop a class of measures of economic segregation that measure what is generally termed the "evenness" dimension of segregation. That is, we wish to measure the extent to which households of different income or wealth are evenly distributed among residential locations. Because economic segregation measures can be seen as a class of ordinal segregation measures (because they measure segregation with respect to an ordered variable, albeit one that has an underlying continuous nature), I adapt the criteria used for ordinal segregation measures (Reardon, 2009) to apply to the case of economic segregation.

Before stating the properties of a useful measure of segregation, I define some notation. Let $y$ be a continuous variable measuring income, and let $F$ be the cumulative density function of $y$ in the population of interest, so that $p = F(y)$ is the percentile rank in the population of interest corresponding to income $y$. Let the population be distributed among $N$ neighborhoods, indexed by $n$.[3] One additional piece of terminology will be useful in the following sections. We say that the income distribution in neighborhood $m$ dominates the income distribution in neighborhood $n$ between incomes $y_a$ and $y_c$ (where $y_a < y_c$) if $F_m(y_b) < F_n(y_b)$ for all $y_b$ such that $y_a < y_b < y_c$. In other words, $m$ dominates $n$ over $y_a$ to $y_c$ if there is a greater proportion of the population of $n$ than $m$ at or below each income from $y_a$ to $y_c$.

*Scale Interpretability*

A segregation index is maximized if and only if within each neighborhood $n$, all individuals have the same income $\bar{y}_n$ (there is no variation in income within any neighborhood). A segregation index is minimized if and only if within each neighborhood $n$, the income distribution is identical to that in the population (if and only if $F_n(y) = F(y)$ for all $n$). It is important to note that this definition of maximum segregation does not depend on the overall income distribution in the population, but only on the extent to which neighborhood income distributions differ from the

[3] More generally, $y$ is a continuous variable describing some person or household characteristics (income, wealth, etc.) and $n$ indexes unordered organizational units (neighborhoods, schools, etc.). I will refer to $y$ as measuring income and $n$ as indexing neighborhoods, however, in order to make the exposition more concrete.

overall distribution. Thus, a key feature of this definition of segregation is that it separates income segregation (the sorting of individuals among neighborhoods) from income inequality (the extent to which incomes vary among individuals).

*Organizational Equivalence*

If the populations in neighborhoods $m$ and $n$ have the same income distribution, and $m$ and $n$ are combined, segregation is unchanged. This is analogous to the standard organizational equivalence condition used in nominal measures of segregation (James & Taeuber, 1985; Reardon & Firebaugh, 2002).

*Population Size Invariance*

If the number of individuals in each neighborhood is multiplied by a positive constant but the income density function within each neighborhood is unchanged, segregation is unchanged. The intuition here is simply that the total population counts do not matter, but only the neighborhood income distributions.

*Rank-Preserving Scale Invariance*

Because we would like an income segregation measure to be independent of income inequality, such measures should be invariant under certain types of changes in the income distribution. At a minimum, we might wish a measure to be invariant under multiplicative changes in the income scale. Such a property would ensure, for example, that doubling each household's income does not affect measured segregation. We might also wish the measure to be invariant under additive changes in the income scale—as would occur if each household income increased by a constant amount. A stronger invariance property—and the one I adopt here—is invariance under rank-preserving changes in income. This requires that a change in the income distribution that leaves each household in their same neighborhood and that does not affect the rank-ordering of households in the income distribution will not change segregation levels. Formally, a measure will be rank-preserving scale invariant if it depends only on the sorting of households in relation to their income percentiles $p$, and requires no information regarding actual income amounts.

Rank-preserving scale invariance implies both multiplicative and additive scale invariance. More importantly, it implies that a segregation measure is independent of the level of income inequality, and depends only on the sorting of households among neighborhoods in relation to their income ranks. If, for example, all households remain in the same residential location but the incomes of high-income households grow and the incomes of low-income households decline while each household maintains its rank in the overall income distribution, then a rank-preserving scale invariant measure of segregation will not change. Watson (2009) notes the advantage of measures with this property when examining income sorting processes.

It is worth noting that our desire for a measure of income segregation that is independent of the income distribution means that we are interested in the sorting of *households* among neighborhoods rather than the sorting of *income* among neighborhoods. That is, we are implicitly treating income as an ordered—but not necessarily interval-scaled—characteristic of households, and we are interested in the extent to which households of different incomes are unevenly distributed among neighborhoods. Any measure that treats income as ordinal- rather than interval-scaled will necessarily be impervious to any rank-preserving changes in the income distribution, by definition.

*Exchanges*

If the income distribution in neighborhood $m$ dominates that in neighborhood $n$ over incomes $y_a$ to $y_b$ (where $y_a < y_b$), and if an individual with income $y_a$ moves from $n$ to $m$ while an individual with income $y_b$ moves from $m$ to $n$, then segregation is reduced. Likewise, if the income of an individual in neighborhood $m$ decreases from $y_b$ to $y_a$, while the income of an individual in neighborhood $n$ increases from $y_a$ to $y_b$, then segregation is reduced. The intuition here is that if we exchange individuals of different incomes in a way that makes the income distributions in two neighborhoods more similar to one another, segregation should be reduced. For example, if neighborhood $m$ has a lower proportion of its population at or below each income level than neighborhood $n$, and if a high-income individual in neighborhood $m$ swaps houses with a lower-income individual in $n$, then segregation should be reduced—because we have raised income levels in $n$ and lowered them in $m$. Reardon (2009) provides a discussion of this in the ordinal case.

*Ordered Exchanges*

If the income distribution in neighborhood $m$ dominates that in neighborhood $n$ over incomes $y_a$ to $y_c$ (with $y_a < y_b < y_c$), and if an individual of income $y_a$ moves from neighborhood $n$ to $m$ while an individual of income $y_c$ moves from $m$ to $n$, then the resulting reduction in segregation will be greater than that resulting if an individual of income $y_a$ moves from neighborhood $n$ to $m$ while an individual of income $y_b$ moves from neighborhood $m$ to $n$ or if an individual of income $y_b$ moves from neighborhood $n$ to $m$ while an individual of income $y_c$ moves from neighborhood $m$ to $n$. Likewise, if the income of an individual in neighborhood $m$ decreases from $y_c$ to $y_a$, while the income of an individual in neighborhood $n$ increases from $y_a$ to $y_c$, then the resulting reduction in segregation will be greater than that resulting if the income of an individual in neighborhood $m$ decreases from $y_b$ to $y_a$, while the income of an individual in neighborhood $n$ increases from $y_a$ to $y_b$ (or if the income of an individual in neighborhood $m$ decreases from $y_c$ to $y_b$, while the income of an individual in neighborhood $n$ increases from $y_b$ to $y_c$). The intuition here is that the effect of an exchange of individuals between neighborhoods or of an exchange of income between individuals ought to be sensitive to the difference in income involved. Exchanges involving individuals whose incomes are farther apart should change segregation more than exchanges of individuals whose incomes are closer to one another's. Reardon (2009) provides some discussion of this in the ordinal case.

In addition to these criteria, there are a number of other features that are desirable—if not always necessary—in a measure of economic segregation. First, a measure should be *additively decomposable*, in the sense described by Reardon and Firebaugh (2002). This enables the partitioning of segregation into between- and within-cluster components (for example, it would allow us to decompose income segregation into between-city and within-city, between neighborhood components). Second, an economic segregation measure should be *spatially adaptable*, meaning that the measure can be adapted to account for the spatial patterning of households. Third, because most sources of data on income and wealth report income or wealth in a set of ordered categories, a useful income segregation measure should be easily computed from categorical data and should be *income category threshold invariant*—that is, it should be insensitive to the set of income thresholds used to define the reported income categories.

**3. Measuring Segregation by an Ordinal Category**

Because income data are generally reported as counts by ordered income category, I begin by describing the approach to measuring segregation among groups defined by ordinal categories developed in Reardon (2009). This approach, I later show, can be readily extended to define measures of segregation by a continuous variable.

Again, it is useful to begin by defining some notation. Let $y$ be an ordinal measure of income, taking on $K$ distinct ordered values $1, 2, \ldots, K$. The set of values of $y$ in a population can be summarized by the $[K-1]$-tuple of cumulative proportions $C = (p_1, p_2, .., p_{K-1})$, where $p_k$ is the proportion of the population with values of $y \leq k$ ($p_K = 1$ by definition, so is not needed to characterize the distribution of $y$). Let $v = g(p_1, p_2, \ldots, p_{K-1})$ denote a measure of the variation in $y$

in the population that can be computed from the set of cumulative proportions. It will be useful to require that $g$ be a concave down function of the $p_k$'s in the $K-1$ dimensional unit square. Let $n \in (1,2,\ldots,N)$ index a set of unordered organizational units (e.g., neighborhoods or census tracts).

Reardon (2009) notes that one way of constructing a segregation measure is to think of it as a form of a variation ratio, where segregation is the proportion of the total variation in a population that is due to differences in population composition of different organizational units (e.g., census tracts). This approach underlies the variation-ratio segregation measures described above. Under this approach, we can define an ordinal segregation measure $\Lambda$ as follows:

$$\Lambda = \sum_{n=1}^{N} \frac{t_n}{T\nu}(\nu - \nu_n) = 1 - \sum_{n=1}^{N} \frac{t_n \nu_n}{T\nu}, \tag{1}$$

where $T$ and $t_n$ are the total population and the population count in unit $n$, respectively, and where $\nu$ and $\nu_n$ are the ordinal variation in $y$ in the population and in unit $n$, respectively.

The key to defining such a measure of ordinal segregation, then, is in defining an appropriate measure of ordinal variation. Reardon (2009) notes that variation in an ordinal variable $y$ is maximized when half the population has the lowest possible value of $y$ and half has the highest possible value of $y$, corresponding to $C = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$ (Berry & Mielke, 1992a, 1992b; Blair & Lacy, 1996, 2000; Kvålseth, 1995a, 1995b). Variation in an ordinal variable $y$ is minimized when all observations have the same value of $y$, corresponding to the $K$ cases of the pattern $C = (0,0,\ldots,0,1,\ldots,1)$, implying that variation in an ordinal variable can be measured by constructing a function that increases as each $p_k$ approaches $\frac{1}{2}$ and decreases as each $p_k$ approaches 0 or 1. One set of such functions are those that have the form

$$v = g(p_1, p_2, \ldots, p_{K-1}) = \frac{1}{K-1} \sum_{k}^{K-1} f(p_k), \tag{2}$$

where $f(p)$ is maximized at $f\left(\frac{1}{2}\right) = 1$ and minimized on the interval [0,1] at $f(0) = f(1) = 0$. If $f$ is twice-differentiable, with a negative second derivative, everywhere on the unit interval, then $g$ is also concave down everywhere on the $K-1$ dimensional unit square.

Reardon (2009, p. 143) shows that an ordinal segregation measure defined as in Equation (1) can be written as the weighted average of $K-1$ binary (non-ordinal) segregation indices. That is, suppose we collapse the ordinal variable $y$ into two categories (those with $y \leq k$ and those with $y > k$). Let $v_k$ denote the variation in this collapsed variable. From Equation (2), we have $v_k = f(p_k)$. Now we define $\Lambda_k$ as the segregation between these two groups (using $v_k = f(p_k)$ in Equation (2) in place of $v$). Reardon shows that Equation (1) can be written as

$$\Lambda = \sum_{k=1}^{K-1} w_k \Lambda_k, \tag{3}$$

where $w_k = v_k / \sum_{j=1}^{K-1} v_j = f(p_k) / \sum_{j=1}^{K-1} f(p_j)$ (see Appendix A1 for derivation). Note that these weights sum to 1. By construction, the weights are maximized when $p_k = \frac{1}{2}$, that is, when dividing the population at the top of category $k$ divides the population in half. Likewise the weights are smallest when $p_k$ is close to 0 or 1, when dividing the population at category $k$ splits the population most unevenly. I return to a discussion of the interpretation of these weights later.

Note that the above method of constructing an ordinal segregation measure produces measures that have features of both the category-based measures of income segregation and the variation ratio measures of income segregation described above. Equation (1) defines a class of variation ratio measures, where segregation is high if the average within-unit variation in $y$ is much smaller than the population variation in $y$, and vice versa. Equation (3), however, shows that this same class of measures can be written as a weighted average of a set of pairwise, category-based measures of segregation. This approach has some intuitive appeal and consistency with much of the prior literature.

Nonetheless, the ordinal measures also share the flaws of the categorical measures, since Equation (3) makes clear that they depend explicitly on the values of the $K$-1 income thresholds used to define the income categories (except in the special case where $\Lambda_k$ is constant across all possible income thresholds, an unlikely scenario). Moreover, Equation (2) makes clear that they share the flaws of the variation ratio measures, since the measure of ordinal variation $v$ on which they are based is not necessarily invariant under changes in the income distribution (unless the thresholds defining the income categories change as well in such a way that they lie at the same percentiles of the income distribution).

*Visualizing the Ordinal Segregation Measures*

The fact that the ordinal segregation indices can be written as weighted averages of a set of pairwise segregation indices enables us to better visualize what the indices measure. To illustrate the measures, consider Figures 1 and 2, which show the cumulative household income percentile density curves for random samples of 50 census tracts in San Francisco County, CA (whose boundaries are identical to those of the city of San Francisco) and in Wayne County, MI (which includes Detroit) in 2000.[4] In both figures, the $x$-axis indicates both the local (i.e., San Francisco or Wayne County) income percentiles and the 15 income thresholds used in the 2000 census (indicated by the vertical dashed lines). Note that the income distribution in San Francisco is shifted right in comparison to Wayne County—25% of households in San Francisco reported incomes greater than $100,000, compared to 12% of Wayne County households. If there were no income segregation in either county, each tract's cumulative household income percentile density curve would fall exactly on the 45-degree line (the heavy black line in each figure). If there were complete income segregation, each tract's curve would be a vertical line at some income level, indicating that within each tract all households have the same income. Thus, one way of thinking

[4] Household income data (for the year 1999) are obtained from Table P52 in SF3 from the 2000 Census. I show only a random sample of tract cumulative income distributions to avoid cluttering the figures.

about a segregation measure is that it should measure the average deviation of the tract cumulative household income percentile density curves from their regional average (which is, by definition, the 45-degree line). By this criterion, Wayne County appears more segregated by income than San Francisco, since the variation of the tract cumulative density curves around the 45-degree line is greater in Wayne County (though admittedly such a comparison is hard to make visually, given the density of the figures).

Figures 1 and 2 here

The ordinal segregation indices, when applied to income data, can be seen as measures of the variation of the tract cumulative household income percentile curves around the 45-degree line. At each income threshold reported in the Census, we know the value of each tract's cumulative household income density curve—that is, we know the percentage of households with incomes above and below each threshold (denoted $p_k$ above). These data can be used to compute the pairwise segregation between households with incomes above and below that threshold (what denoted $\Lambda_k$ above). Equation (3) makes clear that an ordinal segregation measure of the class defined by Reardon (2009) can be seen as a weighted average of these 15 measures, where the weights are maximized when $p_k = \frac{1}{2}$.

A close look at Figures 1 and 2 suggests that the ordinal segregation measures do not fully solve the problems of existing measures of income segregation. Because of the differences in the overall income distributions in the two counties, the Census-defined income thresholds do not fall at the same percentiles of the two local income distributions. Thus, in San Francisco, the ordinal segregation measures are based more heavily on information about the segregation at thresholds in the 10th-50th percentile range (where 9 of 15 thresholds fall) than in the 50th-90th percentile range (where only 5 of 15 thresholds fall). In Wayne County, in contrast, three of the thresholds fall in the 90th-99th percentile range, and only 7 fall in the 10th-50th percentile range. As a result, the ordinal segregation measured in San Francisco is not exactly comparable to that measured in Wayne

County, because of differences in the underlying income distributions. Put differently, the measures depend on the choice of thresholds—a different set of income thresholds would yield different measured levels of segregation. And finally, the measures are also clearly not invariant under changes in income that preserve the shape of the income distribution—a doubling of each household's income would have the effect of moving the thresholds to the left on the figure, meaning that the computed ordinal segregation would depend much more on segregation levels at the low end of the percentile distribution.

## 4. Measures of Rank-Order Segregation

*A General Class of Rank-Order Segregation Measures*

The foregoing discussion illustrates that the ordinal segregation measures do not avoid the flaws of many existing approaches to measuring income segregation. They do improve on existing categorical measures, however, to the extent that they rely on multiple, relatively evenly-spaced thresholds. Moreover, they may be useful measures when measuring segregation by some truly ordinal variable, where the thresholds have some substantive meaning (rather than a variable that is inherently continuous, but measured ordinally, like income). Most importantly for our purposes, they provide the intuition for a related set of segregation measures that is free of their flaws.

Equation (3) describes an ordinal segregation measure as a weighted average of a finite set of $K-1$ pairwise segregation indicies. If we let the number of categories grow arbitrarily large, we can—with some abuse of notation[5]—define a rank-ordered segregation index as

$$\Lambda^R = \int_0^1 \frac{f(p)}{\int_0^1 f(q)dq} \Lambda(p)dp \tag{4}$$

[5] Equation (4) and subsequent equations treat $p$ as if it could take on any of an infinite number of values between 0 and 1. In fact, in a finite population, the number of possible income categories is limited by the number of distinct values realized in the population. However, as long as the population is large, this limitation has no practical effect on the computation of the measures. I use the integral notation here for simplicity.

where $f(p)$ is—as above—a measure of variation between those with incomes above or below the $100 \times p^{th}$ percentile of the income distribution, and $\Lambda(p)$ is the segregation between the same two groups. Equation (4) is simply the limit of the expression in Equation (3) as the number of categories goes to infinity. In principle, such a measure would be free from the problems inherent in the ordinal segregation measures, all of which stem from the finite number of unevenly-spaced income category thresholds.

One additional feature of Equation (4) is worth noting. The definition of $\Lambda(p)$ makes clear that we can, in principle, consider the segregation between any two groups defined by whether they have incomes above or below the $100 \times p^{th}$ percentile of the income distribution. That is, we might be interested in the segregation of the top 10 percent of earners from the bottom 90 percent (what Reardon and Bischoff (2011) term the "segregation of affluence"), or the segregation of the bottom 10 percent from the top 90 percent (the "segregation of poverty"), or the segregation of below-median earners from above-median earners, and so on. For a variety of substantive reasons, the extent of segregation of affluence may differ from the segregation of poverty, and patterns, trends, and reasons for such difference may be of interest in their own right. For example, Reardon and Bischoff (2011) find that the rise in income inequality from 1970-2000, which was driven by growth at the top of the income distribution, led to increased segregation of the affluent, but not of the poor. Thus, the ability to compare groups at any percentile of the income distribution allows for a more refined examination of patterns and trends when the research question warrants it. In other cases, focusing on a summary measure of rank-order segregation, $\Lambda^R$, may be of particular interest. Equation (4) makes clear that the summary measure can be understood as a weighted sum of the threshold-specific measures, a property that enables a consistent type of analysis of segregation regardless of whether our interest is in segregation of a specific part of the income distribution or in the total extent of income segregation.

*Three Measures of Rank-Order Segregation*

Equation (4) defines a general class of measures of rank-order segregation. To define a specific measure, however, we must define a function $f$ that satisfies the conditions described above following Equation (2). Specifically, $f$ should be a concave down function on the unit interval, maximized at $f\left(\frac{1}{2}\right) = 1$ and minimized at $f(0) = f(1) = 0$. Reardon shows that three such possible functions $f$ (each of which yields a useful measure of ordinal segregation) are:[6]

$$E(p) = -[p\log_2 p + (1-p)\log_2(1-p)$$
$$I(p) = 4p(1-p)$$
$$V(p) = 2\sqrt{p(1-p)}. \tag{5}$$

Each of these functions meets the conditions specified above.[7] Moreover, each yields a familiar pairwise segregation index when $K = 2$. Specifically, if we define $f(p) = E(p)$, then

$$\Lambda_k = \sum_{n=1}^{N} \frac{t_n}{TE(p_k)}[E(p_k) - E(p_{nk})] = H_k = H(p_k) \tag{6}$$

is the information theory segregation index (James & Taeuber, 1985; Theil, 1972; Theil & Finezza, 1971) between those with incomes at or below income category $k$ and those with incomes above category $k$. Likewise, if we define $f(p) = I(p)$, then

$$\Lambda_k = \sum_{n=1}^{N} \frac{t_n}{TI(p_k)}[I(p_k) - I(p_{nk})] = R_k = R(p_k) \tag{7}$$

[6] Note we define $0 \cdot \log_2(0) = \lim_{p\to 0^+} p\log_2(p) = 0$. Reardon (2009) identifies a fourth function, $f_4(p) = 1 - |2p - 1|$, but shows that it does not yield a satisfactory measure of segregation (because it is not a concave down function of $p$).

[7] The maxima and minima conditions are straightforward to verify. The second derivatives of the three functions are, respectively, $\frac{d^2E}{dp^2} = \frac{-1}{p(1-p)}$; $\frac{d^2I}{dp^2} = -8$; and $\frac{d^2V}{dp^2} = \frac{-1}{2[p(1-p)]^{3/2}}$. Each of these second derivatives are negative everywhere in the interval (0,1).

is the variance ratio segregation index (Bell, 1954; Coleman, Hoffer, & Kilgore, 1982; Duncan & Duncan, 1955; Zoloth, 1976). Finally, if we define $f(p) = V(p)$, then

$$\Lambda_k = \sum_{n=1}^{N} \frac{t_n}{TV(p_k)} [V(p_k) - V(p_{nk})] = S_k = S(p_k) \tag{8}$$

is Hutchens' square root segregation index (Hutchens, 2001, 2004).

Substituting each of these into Equation (4) defines three rank-order segregation measures: the *rank-order information theory index* ($H^R$):[8]

$$\begin{aligned} H^R &= \int_0^1 \frac{E(p)}{\int_0^1 E(q)dq} H(p)dp \\ &= 2\ln(2) \int_0^1 E(p)H(p)dp\,; \end{aligned} \tag{9}$$

the r*ank-order variance ratio index* ($R^R$):

$$\begin{aligned} R^R &= \int_0^1 \frac{I(p)}{\int_0^1 I(q)dq} R(p)dp \\ &= \frac{3}{2} \int_0^1 I(p)R(p)dp\,; \end{aligned} \tag{10}$$

and the *rank-order square root index* ($S^R$):

$$\begin{aligned} S^R &= \int_0^1 \frac{V(p)}{\int_0^1 V(q)dq} S(p)dp \\ &= \frac{4}{\pi} \int_0^1 V(p)S(p)dp. \end{aligned} \tag{11}$$

Intuitively, $H^R$, $R^R$, and $S^R$ are extensions of the ordinal segregation indices defined in Reardon

[8] See Appendix A2 for calculation of the integrals in the denominator of each of Equations (9)-(11).

(2009) to the case where we have an arbitrarily large number of income categories.

*Visualizing the Rank-Order Segregation Measures*

Equations (9)-(11) define three rank-order segregation measures as weighted averages of the values of pairwise segregation measures at every point in the income distribution. Figures 3-5 allow us to visualize this weighted average. Each figure shows the pairwise household income segregation (as measured by $H$, $R$, or $S$, respectively) computed at each of the 15 Census 2000 thresholds for San Francisco and Wayne County; these enable us to visualize what the complete functions $H(p)$, $R(p)$, and $S(p)$ would look like if we could observe them for each value of $p \in (0,1)$. In addition, each figure illustrates the relative weight (dashed lines) that the pairwise segregation computed at each threshold is given in the calculation of the segregation measures. The weights are displayed on the same scale in the three figures, allowing comparison of their relative magnitude. The weight $I(p)$ used in computation of the rank-order variance ratio segregation index (Figure 4) weights segregation between those above and below the median income relatively more than do the other two weights; the weight $V(p)$ used in the rank-order square root index (Figure 5) weights segregation at the extremes of the income distribution relatively more than the other two. I return to a discussion of the weights later.

Figures 3-5 here

The segregation profiles shown in Figures 3-5 are informative. First, note that segregation, as measured by $H$, $R$, or $S$, is relatively flat across most of the middle of the income percentile distribution in both places, but increases or decreases sharply at the extremes of the distribution, depending on which measure is used. Second, note that, as expected, measured segregation at each income percentile is generally higher in Wayne County than in San Francisco, regardless of which measure is used; the difference is more pronounced at the high end of the income distribution for $H$ and $R$, though not for $S$. Finally, the shapes of the segregation curves differ slightly between San

Francisco and Wayne County: in San Francisco, for example, segregation between those with income above and below the 25th percentile is higher than segregation between those with incomes above and below the 75th percentile, while the opposite is true in Wayne County.

*Computing the Rank-Order Segregation Measures*

To compute the rank-order segregation indices, we must evaluate equations (9)-(11). The formulas for $E(p)$, $I(p)$, and $V(p)$ are defined in Equation (5). Thus, if we knew the functions $H(p)$, $R(p)$, and $S(p)$ on the interval (0,1), we could compute the rank-order segregation measures without relying on an arbitrary set of thresholds. In general, we do not know these functions, but we can estimate them from the $K-1$ values of $H(p_k)$, $R(p_k)$, and $S(p_k)$ that we can measure.

We can estimate the functions $H(p)$, $R(p)$, and $S(p)$ on the interval (0,1) as follows. For each threshold $k \in (1,2,\ldots,K-1)$, we compute $H_k$, $R_k$, or $S_k$ and then plot it against the corresponding $p_k$, the cumulative proportions of the population with incomes equal to or below the threshold $k$, as shown in Figures 3-5. We then fit a polynomial of some order $M \leq K-2$ to the observed points, using weighted least squares (WLS) regression and weighting each point by $E_k^2$, $I_k^2$, or $V_k^2$ (depending on whether we are fitting polynomial $H(p)$, $R(p)$, or $S(p)$, respectively). Weighting the regression by the square of the weight minimizes the weighted squared errors and ensures that the fitted polynomial will fit best for $p_k$ near $\frac{1}{2}$, where $H_k$, $R_k$, or $S_k$ is weighted most.

Given a sufficient number of income categories that are spread widely across the percentiles of the income distribution, we can obtain relatively precise estimates of $H(p)$, $R(p)$, and $S(p)$. We can use the uncertainty in the estimated polynomial to provide information on the amount of uncertainty in the rank-order segregation that arises from the fact that we must estimate $H(p)$, $R(p)$, or $S(p)$. Moreover, to the extent that we estimate $H(p)$, $R(p)$, and $S(p)$ well on the interval (0,1) from the observed points, our estimate of segregation will not be biased by the choice of thresholds we have available. In general, we will not have information on the shape of $H(p)$, $R(p)$, and $S(p)$ at the extreme ends of the income distribution (at points below and above the bottom and top thresholds), except via extrapolation. We can, however, assess the sensitivity of our estimates of $H^R$, $R^R$, and $S^R$ to alternative assumptions about the shape of $H(p)$, $R(p)$, and $S(p)$ by assuming a range of possible shapes of the functional form. In general, the estimates of $H^R$, $R^R$, and $S^R$ will be relatively insensitive to assumptions about $H(p)$, $R(p)$, and $S(p)$ at the ends of the income percentile distribution, because little weight is given to $H(p)$, $R(p)$, and $S(p)$ when $p$ is near 0 or 1.[9]

More specifically, we fit the WLS regression model

$$\Lambda_k = \beta_0 + \beta_1 p_k + \beta_2 p_k^2 + \cdots + \beta_M p_k^M + e_k, \qquad e_k \sim N\left(0, \frac{1}{v_k^2}\right) \tag{12}$$

to the $K-1$ points $(\Lambda_k, p_k)$. This yields the $(M+1) \times 1$ vector of estimated coefficients $\widehat{\mathbf{B}} = \left(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_M\right)'$. In Appendix A3, I show that the segregation indices can be estimated by[10]

$$\widehat{\Lambda}^R = \mathbf{\Delta}'\widehat{\mathbf{B}}, \tag{13}$$

where $\mathbf{\Delta} = (\delta_0, \delta_1, \ldots, \delta_M)'$ is an $(M+1) \times 1$ vector of scalars, such that

$$\delta_m = \int_0^1 \frac{f(p) \cdot p^m}{\int_0^1 f(q)dq} dp. \tag{14}$$

In Appendix A4, I evaluate the integrals in Equation (14) to derive closed-form expressions for $\mathbf{\Delta}$ for each of the three variation functions described in Equation (5), allowing us to use Equation (13) to estimate each of the three rank-order segregation indices from data that provides only ordinal income measures (such as the US Census and many surveys). Values of the $\delta_m$'s for each of the

[9] When weighting by $E$, the bottom and top deciles together carry only 7.5% of the total weight, for example; when weighting by $I$, the bottom and top deciles together carry only 5.6% of the total weight; when weighting by $V$ they together carry 10.4% of the weight, so even if our estimates of $H(p)$, $R(p)$, and $S(p)$ were off by a large amount in the tails of the income percentile distribution, this error would contribute little error to the estimates of $H^R$, $R^R$, and $S^R$.

[10] Fitting the regression model in (12) will also yield the estimated variance-covariance matrix of $\widehat{\mathbf{B}}$, denoted $\mathbf{V}$. The variance in the estimate of $\widehat{\Lambda}^R$ can be computed as $Var\left(\widehat{\Lambda}^R\right) = \mathbf{\Delta}'\mathbf{V}\mathbf{\Delta}$. This provides an estimate of the uncertainty in $\widehat{\Lambda}^R$ that arises from the fact that the polynomial in (12) does not fit the data exactly.

three indices are shown in Table 1.

Table 1 here

*Interpretation of the Rank-Order Segregation Measures*

I have defined three income segregation measures: the rank-order information theory index ($H^R$); the rank-order variance ratio index ($R^R$); and the rank-order square root index ($S^R$). These can each be interpreted in three ways: 1) as weighted averages of pairwise segregation indices (as in Equation (4)); 2) as proportions of the variation in income that lies between rather than within neighborhoods (as in Equation (1); and 3) as measures of the extent to which the cumulative income percentile density curves vary around their mean (as shown in Figures 1 and 2).

First, the interpretation of the rank-order indices as weighted averages of pairwise indices follows directly from Equation (4). In practical terms, given an income-ranked population of $N$ individuals (with no rank ties), we can define $N$-1 thresholds that each dichotomize the population into those with ranks above and below the given threshold. Then the rank-order indices are weighted averages of the $N$-1 values of their corresponding pairwise index obtained by computing the segregation between each of the pairs of groups defined by the thresholds. As shown in Figures 3-5, the weights $E_k$, $I_k$, and $V_k$ have their maxima at $p_k = 0.5$, and their minima at $p_k = 0$ and $p_k = 1$. Thus the measures weight segregation between groups defined by the median of the income distribution most heavily, and segregation between the extreme income groups and the remainder of the population least heavily. Intuitively, this makes sense, because a segregation level computed between those above and below the 99th percentile, for example, tells us very little about the segregation between two randomly chosen individuals, while segregation between those above and below the median income tells us more about overall income segregation.

The weight $I_k$, in particular, has an appealing interpretation. For a given threshold $k$, the probability that two randomly-selected individuals from the population will have incomes on opposite sides of threshold $k$ is $2p_k(1 - p_k)$, which is proportional to $I_k$. Since the segregation level describes the extent of segregation between individuals on either side of the income threshold $k$, we can interpret Equation (6) as a weighted average of the segregation across each threshold, where the value at each threshold is weighted by how informative segregation measured at that threshold is for a randomly chosen pair of individuals.

Second, the rank-order income segregation measures can be interpreted as variation ratios, as in Equation (1). To see this, consider Equation (3), which defines the variation in an ordinal variable as the average, across $K-1$ thresholds, of the function $f(p_k)$, where $p_k$ is the proportion of the population with income ranks below $p_k$. Extending this to the case where the number of thresholds becomes arbitrarily large, the variation in income ranks can be defined as

$$v^R = \int_0^1 f(p_q)dq, \tag{15}$$

where $p_q$ is the proportion of the population below the $100 \times q^{th}$ population percentile (in the total population, $p_q = q$, by definition, but in any given neighborhood, the proportion below any given income rank may differ from $q$). Now, using the same logic as the derivation in Appendix A1, albeit in reverse, Equation (4) can be rewritten as

$$\Lambda^R = \sum_{n=1}^{N} \frac{t_n}{Tv^R}(v^R - v_n^R). \tag{16}$$

Equation (16) shows that the rank-order segregation measures can be written as variation ratios in the same form as Equation (1). The only difference here is that we measure the variation in a (continuous) income *ranks*, rather than (ordinal) income *categories*. Each of the indices measures how much less income rank variation there is within neighborhoods than in the overall population.

Third, the rank-order segregation indices can be interpreted as measures of the extent to which the cumulative income percentile density curves vary around their mean (the 45-degree line). The rank-order information theory index, for example, can be written as

$$H^R = 2\int_0^1 \left\{\sum_{n=1}^{N} \frac{t_n}{T}\left[p_{nq}\ln\left(\frac{p_{nq}}{q}\right) + \left(1-p_{nq}\right)\ln\left(\frac{1-p_{nq}}{1-q}\right)\right]\right\} dq \tag{17}$$

where $p_{nq}$ is the cumulative proportion of those in neighborhood $n$ with incomes at or below percentile 100×$q$. The term inside the brackets is akin to the Theil inequality measure, a measure of the deviation of $p_{nq}$ from its mean ($q$) (Theil, 1967). Averaged over neighborhoods and integrated over $q \in [0,1]$, this yields a measure of the average deviation of the cumulative income percentile density function in neighborhood $n$ from the regional average (the 45-degree line). $H^R$ is therefore a measure of the average variation among neighborhoods of their cumulative income percentile density functions around the 45-degree line.

Likewise, the rank-order relative diversity index can be written as

$$R^R = 6\int_0^1 \left[\sum_{n=1}^{N} \frac{t_n}{T}\left(p_{nq} - q\right)^2\right] dq \tag{18}$$

Since the average of $p_{nq}$ across neighborhoods equals $q$, the term in the brackets is simply the variance of the $p_{nq}$'s at the point on the income distribution given by $q$. Thus, $R^R$ can be interpreted as a measure of the average variance of the neighborhood cumulative percentile density functions.

Note that the rank-order square root index cannot be expressed as a measure of the variation of the neighborhood cumulative percentile density functions, at least not using any traditional measure of variation.[11]

[11] $S^R$ can be written as

$$S^R = \frac{8}{\pi}\int_0^1 \left[\sum_{n=1}^{N} \frac{t_n}{T}\left(\sqrt{q-q^2} - \sqrt{\left(p_{nq} - p_{nq}^2\right)}\right)\right] dq.$$

Although the term in brackets is a measure of the variation in $p_{nq}$ around its mean, it is not a standard

*Incorporating Spatial Proximity Into Measures of Income Segregation*

The income segregation measures developed above do not take into account the spatial patterning of the organizational units (e.g. census tracts) in which income data are collected and reported. Moreover, they are inherently subject to the MAUP, since a different set of definitions of organizational boundaries may yield different computed levels of income segregation. However, because the rank-order segregation measures can be written as weighted averages of traditional pairwise indices, we can define explicitly spatial versions of the rank-order measures simply by substituting spatial versions of the pairwise indices in Equations (9)-(11). Specifically, I define the *spatial rank-order information theory index,*

$$\widetilde{H}^R = 2 \ln 2 \int_0^1 E(p)\widetilde{H}(p)dp\,;$$

(19)

the *spatial rank-order variance ratio index*,

$$\tilde{R}^R = \frac{3}{2}\int_0^1 I(p)\tilde{R}(p)dp\,;$$

(20)

and the *spatial square root index*,

$$\tilde{S}^R = \frac{3}{2}\int_0^1 V(p)\tilde{S}(p)dp\,;$$

(21)

where $\widetilde{H}(p)$, $\tilde{R}(p)$, and $\tilde{S}(p)$ are the pairwise spatial versions of their respective pairwise indices (as defined in Reardon & O'Sullivan, 2004), computed between households with incomes above percentile $100\times p$ and those with incomes at or below percentile $100\times p$.[12] In practice, the functions $\widetilde{H}(p)$, $\tilde{R}(p)$, and $\tilde{S}(p)$ can be estimated from the values of $\widetilde{H}_k$, $\tilde{R}_k$, and $\tilde{S}_k$ computed at each of the $K$-1

measure of variation and has no ready interpretation.

[12] The pairwise spatial segregation indices $\widetilde{H}$ and $\tilde{R}$ can be computed using a program written by Reardon, Matthews, O'Sullivan and colleagues (available at [http://www.pop.psu.edu/services/GIA/research-projects/mss/mss-about](http://www.pop.psu.edu/services/GIA/research-projects/mss/mss-about)).

income thresholds used to define income categories, just as in the aspatial case above. Thus, the spatial versions of the rank-order segregation indices introduce no additional complexity, other than the additional computational burden of computing spatial versions of $\widetilde{H}_k$, $\tilde{R}_k$, and $\tilde{S}_k$.

## 5. Properties of the Rank-Order Income Segregation Measures

Because each of the three indices described above can be written as a weighted average of pairwise indices, they inherit many of the properties of these pairwise indices. The pairwise information theory index $H$, variance ratio index $R$, and square root index $S$ each meet all of the relevant criteria described in section 2 above: scale interpretability; organizational equivalence; population size invariance; exchanges; additive decomposability; and spatial adaptability (Hutchens, 2001, 2004; James & Taeuber, 1985; Reardon & Firebaugh, 2002). Moreover, the ordinal versions of $H$, $R$, and $S$ each satisfy these properties, as well as the property of ordered exchanges (Reardon, 2009). Because the rank-order measures can be seen as ordinal measures with an arbitrarily large number of categories, the proofs in Reardon (2009) that show that the ordinal measures satisfy these properties apply as well to the rank-order measures. Thus, the only two criteria that require further discussion are the rank-preserving scale invariance and income category threshold invariance properties.

*Rank-Preserving Scale Invariance.*

Each of the three measures described above (and any similar measure, such as Watson's CGI (2009)) relies only on information about household's rank in the income distribution (rather than on actual income in monetary units). As a result, any rank-preserving change in the income distribution will not affect the measured segregation. This means that changes in the income distribution that increase or reduce income inequality or changes that raise or lower all households' income (through inflation, for example) will not affect measured segregation so long as the changes do not alter households' ranks in the income distribution.

A consequence of this property is that the rank-order segregation measures separate the measurement of income *segregation* from income *inequality*. Consider a region made up of two neighborhoods, one in which all households have incomes of \$50,000 and one in which all households have incomes of \$55,000. If income is transferred among households so that the first neighborhood now consists of households each with incomes of \$5,000 and the second consists of households each with incomes of \$100,000, the rank-order segregation measures will indicate no change in segregation—in both cases the neighborhoods are sorted similarly by income ranks. However, income inequality will have increased substantially. In this way, the rank-order segregation indices measure the *sorting* dimension of the spatial income distribution, whereas income inequality indices measure the *variation* dimension of the income distribution. In principle, the extent of sorting is independent of the extent of variation. The rank-order measures thus are measures of income sorting. Coupled with measures of income inequality, they provide a more complete description of spatial income distribution than either does alone.

*Income Category Threshold Invariance*

Strictly speaking, the rank-order income segregation measures are not invariant to changes in the locations of the income category thresholds, because the function $\Lambda(p)$ must be estimated from a finite set of points using a (polynomial) parametric function. Both the functional form (e.g., the order of the polynomial) and the locations of the income category thresholds may affect the estimated function $\widehat{\Lambda}(p)$ and so may affect the estimated segregation $\widehat{\Lambda}^R$. However, so long as there are a sufficient number of thresholds that span most of the income distribution, and so long as the fitted polynomial is of high enough order, the estimate of $\Lambda^R$ will not be very sensitive to the location of the thresholds.

To illustrate this, I use the data from San Francisco and Wayne counties described above. I first compute the three rank-order segregation measures using polynomials of order 1-8. I use Equation (13) and the formula in footnote 11 to estimate the segregation level and its standard error (the uncertainty in the estimates arises both because of sampling variation—income data from the Census are based on a sample of households—and because of the misfit of the functional form to the data). Figure 6 reports the estimated segregation levels, based on polynomials of order 1-8, for each of the three measures and for both San Francisco and Wayne counties.

Figure 6 here

Recall that the polynomials are fit to the data depicted in Figures 3-5. It is clear from these figures that polynomials of order 3 or higher are likely necessary to fit the data reasonably well. Indeed, Figure 6 shows that the estimated segregation is relatively imprecise and sensitive to the polynomial order for orders lower than 3 or 4, but for polynomials of order 4 or higher, the estimated segregation is insensitive to the order of the polynomial and is very precisely estimated. The rank-order information theory and variance ratio indices are more precise and more stable than the square root index, likely because the square root index weights segregation values at thresholds at the extremes of the income distribution more heavily than the other two measures. As a result, the estimates are more sensitive to the extrapolation of the fitted polynomial beyond the extreme thresholds.

The fact that the estimated segregation is relatively insensitive to the order of the fitted polynomial beyond order 4 suggests that, at least in these data, we can safely use fourth-order polynomials to estimate segregation. In general, however, it is useful to examine plots like those shown in Figures 3-5 to ensure that the polynomial is of sufficiently high order to capture the segregation function well. It is also useful to check for the robustness of results across different polynomial orders, as done here.

A second concern is with the potential sensitivity of the estimates to differences in the location of the income thresholds reported. Figure 6 above relies on the 15 income thresholds provided in recent Census data. As Figures 3-5 make clear, these thresholds span most of the income range in both counties, and are spaced sufficiently evenly to allow precise estimation of the function $\Lambda(p)$. To investigate how the estimated segregation may depend on the number and range of the income thresholds available, we can drop different thresholds from the data to simulate the condition where we have fewer, and possibly less broadly or evenly spaced, available thresholds. Figure 7 reports the estimated segregation using 8 different subsets of the thresholds. In each case, the segregation measure is computed using a fourth-order polynomial to approximate $\Lambda(p)$.

Figure 7 here

In each panel of Figure 7 are 8 estimates of segregation. The first is based on all the thresholds, for comparison. The second and third estimates use only the even or odd thresholds, respectively. These have only 7 or 8 thresholds instead of the full 15, but they capture most of the range of the full set. In general, the estimated segregation levels from these subsets of thresholds are very similar to—and are similarly precise as—those based on the full sample. The fourth set uses only every third threshold (thresholds 1, 4, 7, 10, 13, and 15) and again yields estimates very close to those based on the full set of thresholds. The fifth and sixth estimates use sets of thresholds that exclude the highest 2 or 4 thresholds, respectively; while the seventh and eighth estimates exclude the lowest 2 or 4 thresholds, respectively. Here there is some evidence that the estimates may be quite sensitive to the set of thresholds used. In particular, the San Francisco estimates are quite sensitive to the exclusion of the top four thresholds, while the Wayne County estimates are quite sensitive to the exclusion of the bottom four thresholds. The reason for this is evident in Figures 3-5. Because the income distribution is much higher in San Francisco than Wayne County, the fifth highest threshold in San Francisco corresponds to the 63$^{rd}$ percentile of the income distribution, while in Wayne County it corresponds to the 78$^{th}$ percentile. Thus, dropping the top 4 thresholds in San Francisco much more severely limits our information about the shape of the segregation function in the upper part of the income distribution than it does in Wayne County. For the same reason, dropping the bottom 4 thresholds affects measured segregation in Wayne County much more than in San Francisco (the 5th threshold is at the 28th percentile in San Francisco and the 38th percentile in Wayne County).

As a rule of thumb, then, the estimates appear relatively insensitive to the location of the thresholds so long as the thresholds span roughly the 25th to 75th percentile range. Moreover, the rank-order information theory and variance ratio indices are slightly less sensitive than the square root index to the absence of thresholds at the extremes, because they weight segregation at the extremes less (see Figures 3-5). Although the results for San Francisco and Wayne counties obviously may not generalize to all places, all income distributions, and all choices of thresholds, they provide reassurance regarding the use of Census income categories for the computation of income segregation in the U.S. The income distributions in San Francisco and Wayne Counties are among the highest and lowest in the U.S., but even in these two counties the income thresholds span almost the entire income range (the 10th-93rd percentiles in San Francisco and the 12th-98th percentiles in Wayne County), suggesting that measured segregation in all metropolitan areas in the U.S. is largely insensitive to the location of the thresholds. Moreover, although the income categories used in the Census as well as the U.S. income distribution have changed over time, the income thresholds have historically covered a roughly similar range of the income distribution.

*Sensitivity to Exchanges*

Although the fact that the ordinal segregation indices satisfy the exchange criteria ensures that the rank-order measures do as well, it is instructive to examine the differential sensitivity of the three indices to exchanges. Consider an exchange $x$ in which a household of income percentile $r$ moves from neighborhood $m$ to $n$ while a household of income percentile $s$ (where $r < s$) moves from neighborhood $n$ to $m$. It is straightforward to show (see Appendix A5) that the derivatives of the three measures with respect to such an exchange can each be written as

$$\frac{d\Lambda^R}{dx} = \frac{1}{T}\int_r^s \left[g\big(F_n(q)\big) - g\big(F_m(q)\big)\right]dq$$

(22)

where, as above, $F_n(q)$ is the proportion of households in neighborhood $n$ with incomes percentiles below $q$, and where $g$ is an increasing function on the interval (0,1). The fact that $g$ is an increasing function implies that the derivative is positive when $F_m(q) < F_n(q)$ for all $q \in (r, s)$; which in turn implies that the indices satisfy the exchange criterion. Equation (22) indicates that segregation between two neighborhoods depends on the differences in a monotonic transformation of their cumulative income percentile density functions. The shape of this monotonic transformation indicates what types of exchanges an index is more or less sensitive to. In particular, the slope of the function $g$ at a given value of $F(q)$ indicates the effect of a disequalizing exchange of households between two neighborhoods that initially have identical proportions of households with incomes below $q$. The top panel of Figure 8 illustrates the function $g$ used in the derivatives of each of the three rank-order indices; the bottom panel illustrations the derivative $\frac{dg}{dF(q)}$ for each of the three functions.

Figure 8 here

Figure 8 shows that $H^R$ and $S^R$ are much more sensitive than $R^R$ to a given difference in income proportions between neighborhoods when the proportions are near 0 or 1. This indirectly implies that $H^R$ and $S^R$ are more sensitive to the segregation of those at the extremes of the income distribution from other households than they are to the segregation of above-median from below-median income households, while $R^R$ is equally sensitive to the segregation across all income thresholds. However, given that there is greater variation in the cumulative income percentile density functions at the middle of the income distribution than the tails (see Figures 1-2), one might argue that a fixed difference in household proportions below an extreme income percentile represents a larger substantive difference between neighborhoods than the same difference at the median income (e.g., the difference between having 45% or 55% of households in a neighborhood below the median income is a smaller difference, in substantive terms, than the difference between having 0% and 10% of households below the 5th percentile of the income distribution). By this argument, $H^R$ and $S^R$ more appropriately respond to exchanges or differences in neighborhood composition than does $R^R$.

## 6. Conclusion

I have proposed three measures of rank-order segregation—the *rank-order information theory index* ($H^R$), the *rank-order variance ratio index* ($R^R$), and the *rank-order square root index* ($S^R$). These indices have several appealing features. First, they are insensitive to rank-preserving changes in income, since the measures are based on the ranks of incomes rather than their numerical values. As a result, the measures are independent of the extent of income inequality and allow comparisons across place and time regardless of the units of income or differences in the cost of living. Second, the measures are relatively easy to calculate, since they require (in the aspatial case) simply computing a series of pairwise segregation values using existing measures of segregation ($H$, $R$ or $S$), fitting a polynomial regression line through these values, and then computing a linear combination of the estimated parameters. When used to compute income segregation, the rank-order measures are largely insensitive to the set of income thresholds that define the categories in which income is reported, so long as the thresholds span most of the range of income percentiles. As a result, they do not require one to make any assumption about the shape of income distributions; in particular, no assumption is needed regarding top-coded incomes, a problem that complicates many other measures of income segregation. Third, the measures are easily adapted to account for spatial proximity, following the approach of Reardon and O'Sullivan (2004). In the spatial case, the computational steps are the same, but the pairwise segregation indices must be computed using some spatially-sensitive method.

Each of the indices I describe satisfies the set of criteria necessary for a useful income segregation measure. However, the three indices differ in several ways. Most importantly, the indices are differently sensitive to segregation at different thresholds of the income distribution. Because $H^R$ and $S^R$ are more sensitive to segregation at the ends of the income distribution, they may be most useful for research that is substantively interested in the residential concentration of poverty or affluence. Of these two, $S^R$ is the most sensitive to segregation at the extremes of the income distribution (and the least sensitive to segregation at the median); indeed the differential sensitivity of $S^R$ to segregation at different parts of the income distribution may be too extreme—it is very insensitive to variation among neighborhoods when the proportions of households at some income level are between 0.2 and 0.8, a rather wide range.

An additional difference among the indices is that $H^R$ and $R^R$ are based on much more widely-used pairwise segregation indices; Hutchens' square root index has not been widely used and so is unfamiliar to most scholars. Moreover, the functional form of $S^R$ and its sensitivity to between-neighborhood differences does not have a readily interpretable form, unlike $H^R$ and $R^R$, which are based in the concepts of entropy and variance, respectively.

Regardless of the differences among them, the great advantage of the rank-order segregation measures is that they measure the extent to which households are sorted by income independently of the extent of income inequality in the population. As a result, they allow for meaningful comparisons regarding the degree of residential sorting by income, across place and time and regardless of changes in the shape of the income distribution, inflation, or cost-of-living. Indeed, they allow the measurement of segregation by any characteristic that has an underlying continuous distribution, such as age, test score, or years of schooling completed. As a result, the measures described here may prove useful in many areas of research, beyond their obvious application to income segregation.

**Appendices**

*A1: Derivation of Equation (3)*

Let $v_k$ indicate the variation of a population that is divided into two groups, where one group consists of all those in income category $k$ or below, and the other groups consists of all those in categories $k+1$ or above. Then, as noted in the text, $v_k = f(p_k)$. Likewise, denote $v_{nk} = f(p_{nk})$, where $p_{nk}$ is the cumulative proportion in category $k$ or lower in unit $n$. Then we can rewrite Equation (1) as:

$$\Lambda = \sum_{n=1}^{N} \frac{t_n}{Tv}(v - v_n)$$

$$= \sum_{n=1}^{N} \frac{t_n}{Tv}\left[\frac{1}{K-1}\sum_{k=1}^{K-1}(v_k - v_{nk})\right]$$

$$= \sum_{k=1}^{K-1} \frac{v_k}{(K-1)v}\left[\sum_{n=1}^{N} \frac{t_n}{Tv_k}(v_k - v_{nk})\right]$$

$$= \sum_{k=1}^{K-1} \frac{v_k}{(K-1)v}\Lambda_k$$

$$= \sum_{k=1}^{K-1} \frac{v_k}{\sum_{j=1}^{K-1} v_j}\Lambda_k.$$

*A2: Derivation of denominators in Equations (9)-(11)*

We wish to evaluate the integral

$$\int_0^1 v(p)dp,$$

where $v(p)$ is the value $f(p)$. For the ordinal information theory index, $v(p) = -[p\log_2 p + (1-p)\log_2(1-p)]$. The integral is then

$$\int_0^1 v(p)dp = -\int_0^1 [p\log_2 p + (1-p)\log_2(1-p)]dp$$

$$= \frac{x + (x-1)^2 \ln(1-\mathrm{x}) - x^2 \ln(x)}{2\ln(2)}\Bigg|_0^1$$

$$= \left(\frac{1}{2\ln 2}\right)[1 + 0 - 0 - 0 - 0 + 0]$$

$$= \frac{1}{2\ln 2}.$$

For the ordinal variation ratio index, $v(p) = 4p(1-p)$. The integral is then

$$\int_0^1 v(p)dp = \int_0^1 [4p(1-p)]dp$$

$$= 2p^2 - \frac{4}{3}p^3\Big|_0^1$$

$$= \frac{2}{3}.$$

For the ordinal square root index, $v(p) = 2\sqrt{p(1-p)}$. The integral is then

$$\int_0^1 v(p)dp = \int_0^1 \left[2\sqrt{p(1-p)}\right] dp$$

$$= \left[-\frac{4}{3}(1-\mathrm{p})^{\frac{3}{2}} \cdot \ {}_2F_1\left(\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{5}{2}, 1-p\right)\right]\Bigg|_0^1$$

where ${}_2F_1(a,b,c,x)$ is Gauss's hypergeometric function. At $x = 0$, ${}_2F_1(a,b,c,x) = 0$; at $x = 1$, Gauss's hypergeometric theorem yields

$${}_2F_1(a,b,c,1) = \frac{\Gamma(c-b-a)\Gamma(c)}{\Gamma(c-b)\Gamma(c-a)},$$

where Γ is the gamma function.[13] Thus, the integral evaluates to

[13] Weisstein, Eric W. "Gauss's Hypergeometric Theorem." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/GausssHypergeometricTheorem.html; Weisstein, Eric W. "Gamma Function." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/GammaFunction.html.

$$\int_0^1 v(p)dp = \left[-\frac{4}{3}(1-\mathrm{p})^{\frac{3}{2}} \cdot \ {}_2F_1\left(\frac{3}{2}, -\frac{1}{2}, \frac{5}{2}, 1-p\right)\right]\Bigg|_0^1$$

$$= \frac{4}{3} \cdot \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)\Gamma(1)}$$

$$= \frac{\pi}{4}.$$

*A3: Derivation of Equations (13)-(14)*

Fitting the model in Equation (12) yields

$$\widehat{\Lambda}(p) = \sum_{m=0}^{M} \hat{\beta}_m \, p^m.$$

Substituting this into Equation (4), rearranging terms, and integrating, yields

$$\widehat{\Lambda}^R = \int_0^1 \frac{f(p)}{\int_0^1 f(q)dq} \widehat{\Lambda}(p)dp$$

$$= \int_0^1 \frac{f(p)}{\int_0^1 f(q)dq} \left[\sum_{m=0}^{M} \hat{\beta}_m p^m\right] dp$$

$$= \sum_{m=0}^{M} \left[\int_0^1 \frac{f(p)p^m}{\int_0^1 f(q)dq} dp\right] \hat{\beta}_m$$

$$= \sum_{m=0}^{M} \delta_m \hat{\beta}_m \, ,$$

where

$$\delta_m = \int_0^1 \frac{f(p)p^m}{\int_0^1 f(q)dq} dp.$$

*A4: Derivation of Formulae in Table 1*

For the rank-order information theory index,

$$\delta_m = 2\ln 2\int_0^1 E(p)p^m dp$$

$$= -2\int_0^1 [p^{m+1}\ln(p) + p^m(1-p)\ln(1-p)]dp$$

$$= -2\int_0^1 \left[p^{m+1}\ln(p) + \left(1-(1-p)\right)^m(1-p)\ln(1-p)\right]dp$$

$$= -2\int_0^1 [p^{m+1}\ln(p)]dp + \int_0^1 \left[\sum_{n=0}^{m}(-1)^{m-n}{}_mC_n(1-p)^{m-n+1}\ln(1-p)\right]dp$$

$$= -2\int_0^1 [p^{m+1}\ln(p)]dp + \sum_{n=0}^{m}(-1)^{m-n}{}_mC_n\int_0^1 [(1-p)^{m-n+1}\ln(1-p)]dp$$

$$= -2\left[p^{m+2}\left(\frac{(m+2)\ln(p)-1}{(m+2)^2}\right)\Bigg|_0^1 - \sum_{n=0}^{m}(-1)^{m-n}{}_mC_n(1-p)^{m-n+2}\left(\frac{(m-n+2)\ln(1-p)-1}{(m-n+2)^2}\right)\Bigg|_0^1\right]$$

$$= \left[\frac{2}{(m+2)^2} + 2\sum_{n=0}^{m}\frac{(-1)^{m-n}{}_mC_n}{(m-n+2)^2}\right],$$

where ${}_mC_n = m!/n!\,(m-n)!$ is the combinatorial function.

For the rank-order variation ratio theory index,

$$\delta_m = \frac{3}{2}\int_0^1 I(p)p^m dp$$

$$= \frac{3}{2}\int_0^1 4p(1-p)p^m dp$$

$$= 6\int_0^1 p^{m+1}(1-p)dp$$

$$= 6\left(\frac{p^{m+2}}{m+2} - \frac{p^{m+3}}{m+3}\right)\Bigg|_0^1$$

$$= \frac{6}{(m+2)(m+3)}.$$

For the rank-order square root index,

$$\delta_m = \frac{4}{\pi}\int_0^1 V(p)p^m dp$$

$$= \frac{4}{\pi}\int_0^1 2\sqrt{p(1-p)}p^m dp$$

$$= \frac{8}{\pi}\int_0^1 p^{m+\frac{1}{2}}\sqrt{1-p}\,dp$$

$$= \frac{8}{\pi}\left[-\tfrac{2}{3}(1-p)^{\frac{3}{2}} \cdot \; {}_2F_1\left(\tfrac{3}{2}, -m-\tfrac{1}{2}, \tfrac{5}{2}, 1-p\right)\right]\Bigg|_0^1$$

where ${}_2F_1(a,b,c,x)$ is Gauss's hypergeometric function (which is equal to 0 for $x$=0). At $x$=1, Gauss's hypergeometric theorem gives

$${}_2F_1\left(\tfrac{3}{2}, -m-\tfrac{1}{2}, \tfrac{5}{2}, 1\right) = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(m+\frac{3}{2}\right)}{\Gamma(1)\Gamma(m+3)},$$

where $\Gamma$ is the gamma function. Thus, the integral evaluates to

$$\delta_m = \frac{8}{\pi}\left[\frac{2}{3}\cdot\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(m+\frac{3}{2}\right)}{\Gamma(1)\Gamma(m+3)}\right]$$

$$= \frac{8}{\pi}\left[\frac{2}{3}\cdot\frac{\frac{\frac{3}{4}\pi\left(1\cdot 3\cdot 5\cdots(2m+1)\right)}{2^{m+1}}}{1\cdot(\mathrm{m}+2)!}\right]$$

$$= 4\left[\frac{\left(1\cdot 3\cdot 5\cdots(2m+1)\right)}{2^{m+1}(m+2)!}\right]$$

$$= 4\prod_{n=0}^{m}\frac{2n+1}{2n+4}.$$

*Appendix A5: Sensitivity of Indices to Exchanges*

Suppose a household of income percentile $r$ moves from neighborhood $m$ to $n$ while a household of income percentile $s$ (where $r < s$) moves from neighborhood $n$ to $m$. Taking the derivative of Equation (4) with respect to such an exchange, and noting that $\frac{d\Lambda(q)}{dx} = 0$ for values of $q < r$ and $q > s$, yields

$$\frac{d\Lambda^R}{dx} = \frac{d}{dx}\left[\int_0^1 \frac{f(q)}{\int_0^1 f(z)dz}\Lambda(q)dq\right]$$

$$= \int_0^1 \frac{f(q)}{\int_0^1 f(z)dz} \cdot \frac{d\Lambda(q)}{dx} dq$$

$$= \int_r^s \frac{f(q)}{\int_0^1 f(z)dz} \cdot \frac{d\Lambda(q)}{dx} dq \, .$$

For the rank-order information theory index, this implies

$$\frac{dH^R}{dx} = 2 \ln 2 \int_0^1 E(q) \frac{dH(q)}{dx} dq$$

James and Taeuber (1985) show that the derivative of $H$ with respect to an exchange is

$$\frac{dH(q)}{dx} = \frac{1}{TE(q)} \log_2 \left( \frac{p_{nq}(1 - p_{mq})}{p_{mq}(1 - p_{nq})} \right),$$

where $p_{mq}$ is the proportion of households with income less than or equal to $q$ in neighborhood $m$ (i.e., $p_{mq} = F_m(q)$). Thus

$$\frac{dH^R}{dx} = \frac{2}{T} \int_r^s \ln\left(\frac{F_n(q)}{1 - F_n(q)}\right) - \ln\left(\frac{F_m(q)}{1 - F_m(q)}\right) dq \, .$$

The term inside the integral is the between-neighborhood difference in the log odds of the proportions of households with income below the $q^{th}$ percentile of the overall income distribution. The change in segregation resulting from an exchange is therefore proportional to average of this difference over the income range from $r$ to $s$.

For the rank-order variance ratio index,

$$\frac{dR^R}{dx} = \frac{3}{2} \int_0^1 I(q) \frac{dR(q)}{dx} dq$$

James and Taeuber (1985) show that the derivative of $R$ with respect to an exchange is

$$\frac{dR(q)}{dx} = \frac{8}{TI(q)}\left(p_{nq} - p_{mq}\right).$$

Thus

$$\frac{dR^R}{dx} = \frac{12}{T}\int_r^s [F_n(q) - F_m(q)]dq$$

The term inside the integral is the between-neighborhood difference in the proportions of households with income below the $q^{th}$ percentile of the overall income distribution. The change in segregation resulting from an exchange is therefore proportional to average of the differences in proportions of households with incomes less than $q$, for $r \leq q \leq s$.

For the rank-order square root index,

$$\frac{dS^R}{dx} = \frac{4}{\pi}\int_0^1 V(q)\frac{dS(q)}{dx}dq$$

It is straightforward to show that the derivative of $S$ with respect to an exchange is

$$\frac{dS(q)}{dx} = \frac{1}{TV(q)}\left[\frac{2p_{nq} - 1}{\sqrt{p_{nq}\left(1 - p_{nq}\right)}} - \frac{2p_{mq} - 1}{\sqrt{p_{mq}\left(1 - p_{mq}\right)}}\right]$$

Thus

$$\frac{dS^R}{dx} = \frac{4}{\pi T}\int_r^s \left[\frac{2F_n(q) - 1}{\sqrt{F_n(q)(1 - F_n(q))}} - \frac{2F_m(q) - 1}{\sqrt{F_m(q)\left(1 - F_m(q)\right)}}\right]dq\,.$$

The term inside the integral has no ready interpretation, though it can be understood as the between-neighborhood difference in a monotonic transformation of the proportions of households with income below the $q^{th}$ percentile of the overall income distribution.

For each of the three indices, then, the change in segregation resulting from an exchange $x$ can be written as

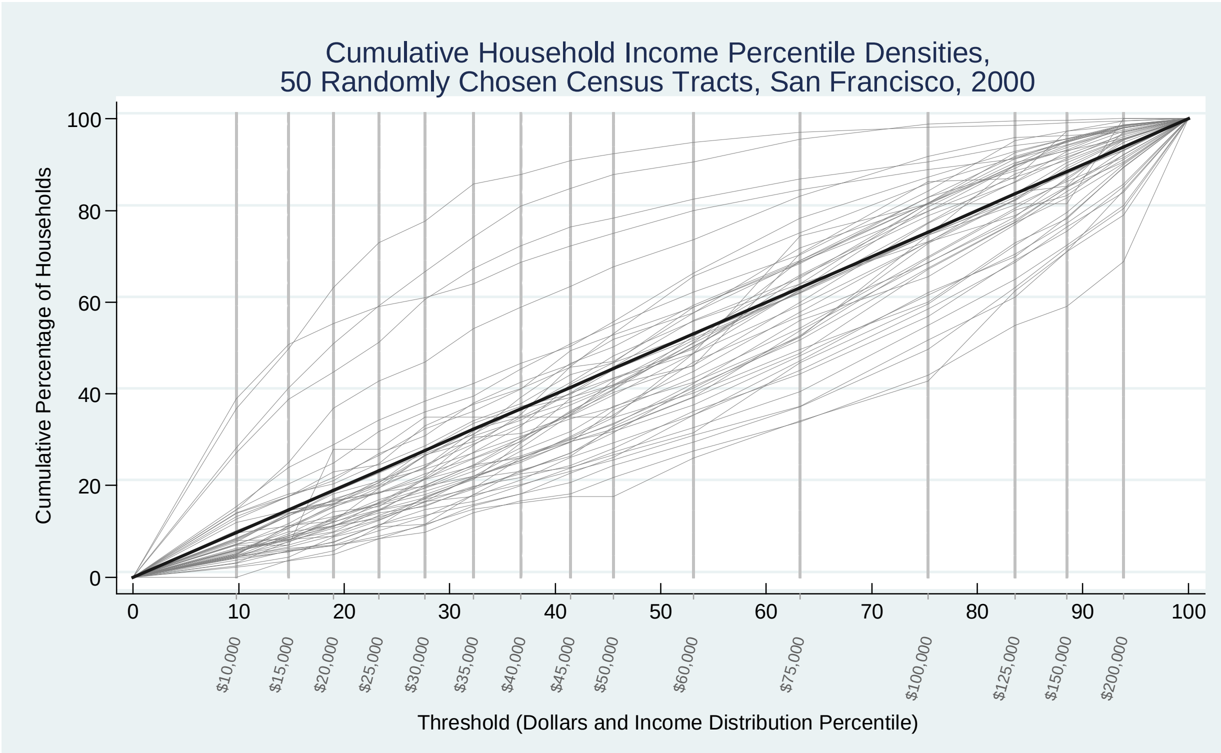
$$\frac{d\Lambda^R}{dx} = \frac{1}{T}\int_r^s \left[g\left(F_n(q)\right) - g\left(F_m(q)\right)\right]dq$$

where $g$ is an increasing function on the interval (0,1). The three functions $g$ and their derivatives are plotted in Figure 8. Because $g$ is increasing, $\left(F_n(q) > F_m(q)\right) \forall q \in (r, s) \Longrightarrow \frac{d\Lambda^R}{dx} > 0$, so each index satisfies the exchange criterion.

**Table 1: Weights for Computing Rank-Order Segregation Indices from Fitted Segregation Profiles**

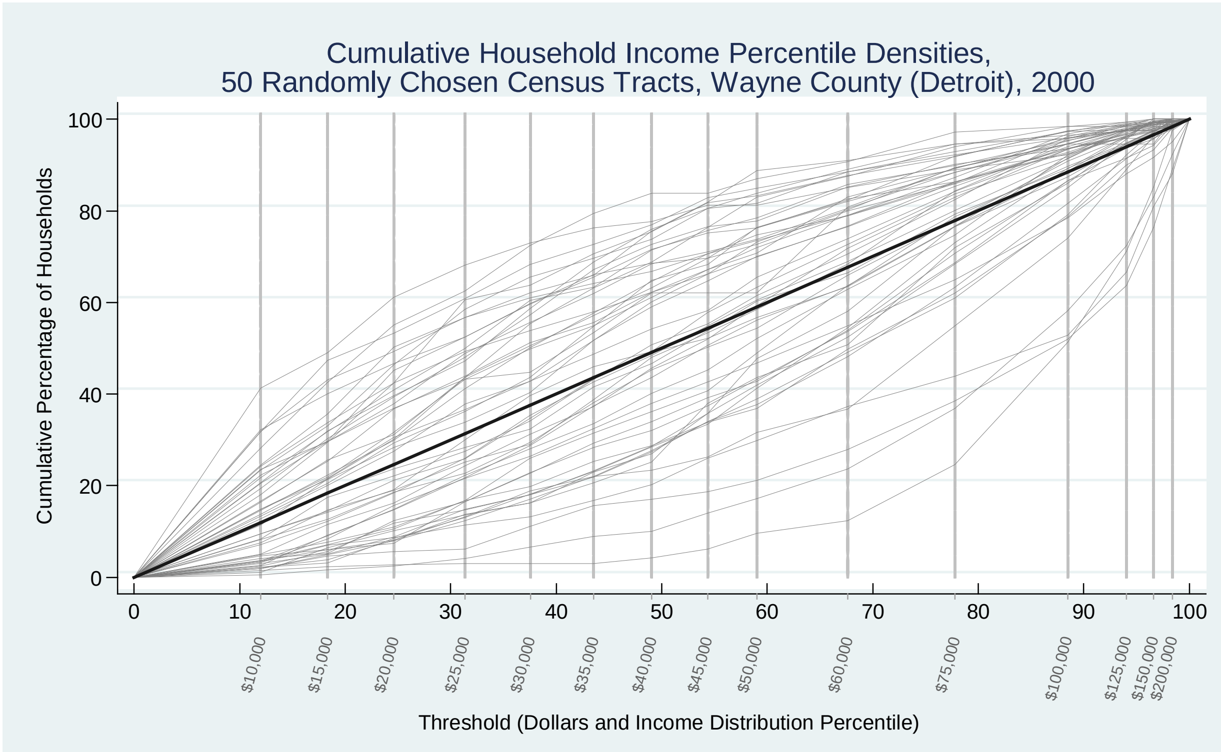
| | Segregation Index | | |
|---|---|---|---|
| | Rank-order information theory index ($H^R$) | Rank-order variation ratio index ($R^R$) | Rank-order square root index ($S^R$) |
| $\delta_0$ | $1$ | $1$ | $1$ |
| $\delta_1$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\delta_2$ | $\frac{11}{36}$ | $\frac{3}{10}$ | $\frac{5}{16}$ |
| $\delta_3$ | $\frac{5}{24}$ | $\frac{1}{5}$ | $\frac{7}{32}$ |
| $\delta_4$ | $\frac{137}{900}$ | $\frac{1}{7}$ | $\frac{21}{128}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\delta_m$ | $\frac{2}{(m+2)^2} + 2\sum_{n=0}^{m} \frac{(-1)^{m-n} {}_mC_n}{(m-n+2)^2}$ | $\frac{6}{(m+2)(m+3)}$ | $4\prod_{n=0}^{m} \frac{2n+1}{2n+4}$ |

**Figure 1**



Cumulative Household Income Percentile Densities,
50 Randomly Chosen Census Tracts, San Francisco, 2000
Cumulative Percentage of Households
0
20
40
60
80
100
0
10
20
30
40
50
60
70
80
90
100
$10,000
$15,000
$20,000
$25,000
$30,000
$35,000
$40,000
$45,000
$50,000
$60,000
$75,000
$100,000
$125,000
$150,000
$200,000
Threshold (Dollars and Income Distribution Percentile)

**Figure 2**



Cumulative Household Income Percentile Densities,
50 Randomly Chosen Census Tracts, Wayne County (Detroit), 2000
Cumulative Percentage of Households
0
20
40
60
80
100
0
10
20
30
40
50
60
70
80
90
100
$10,000
$15,000
$20,000
$25,000
$30,000
$35,000
$40,000
$45,000
$50,000
$60,000
$75,000
$100,000
$125,000
$150,000
$200,000
Threshold (Dollars and Income Distribution Percentile)

**Figure 3**

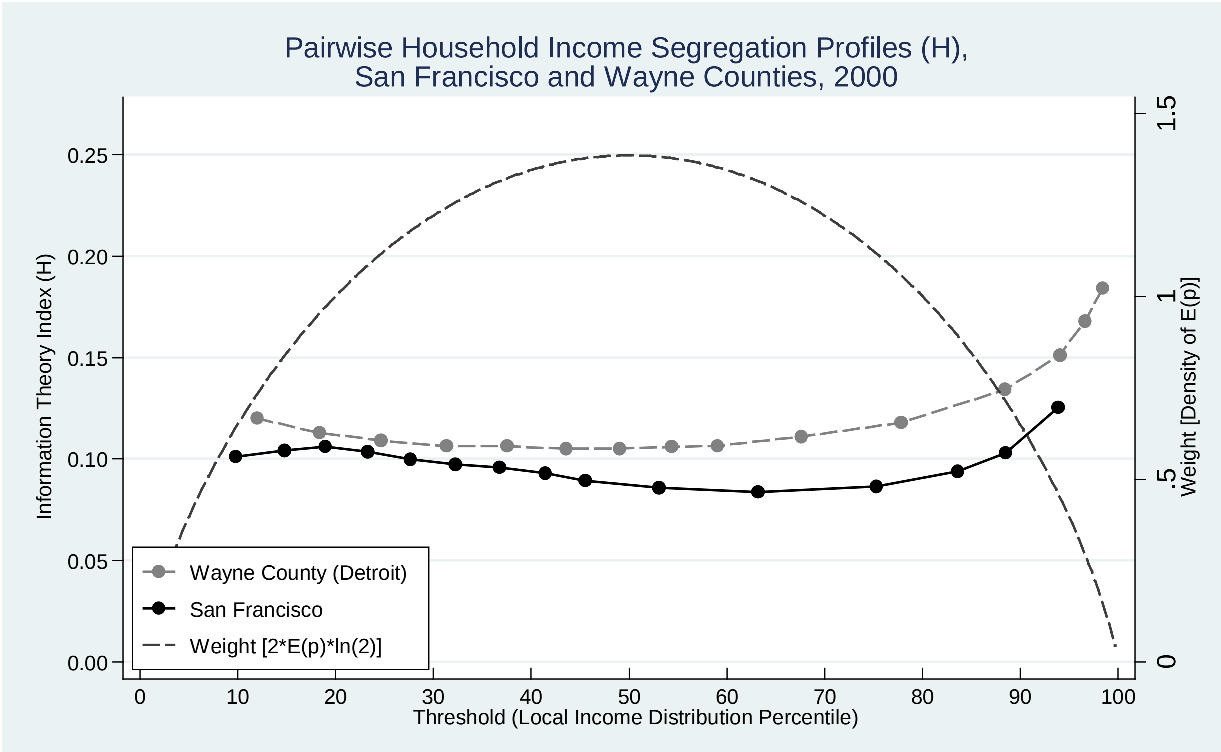
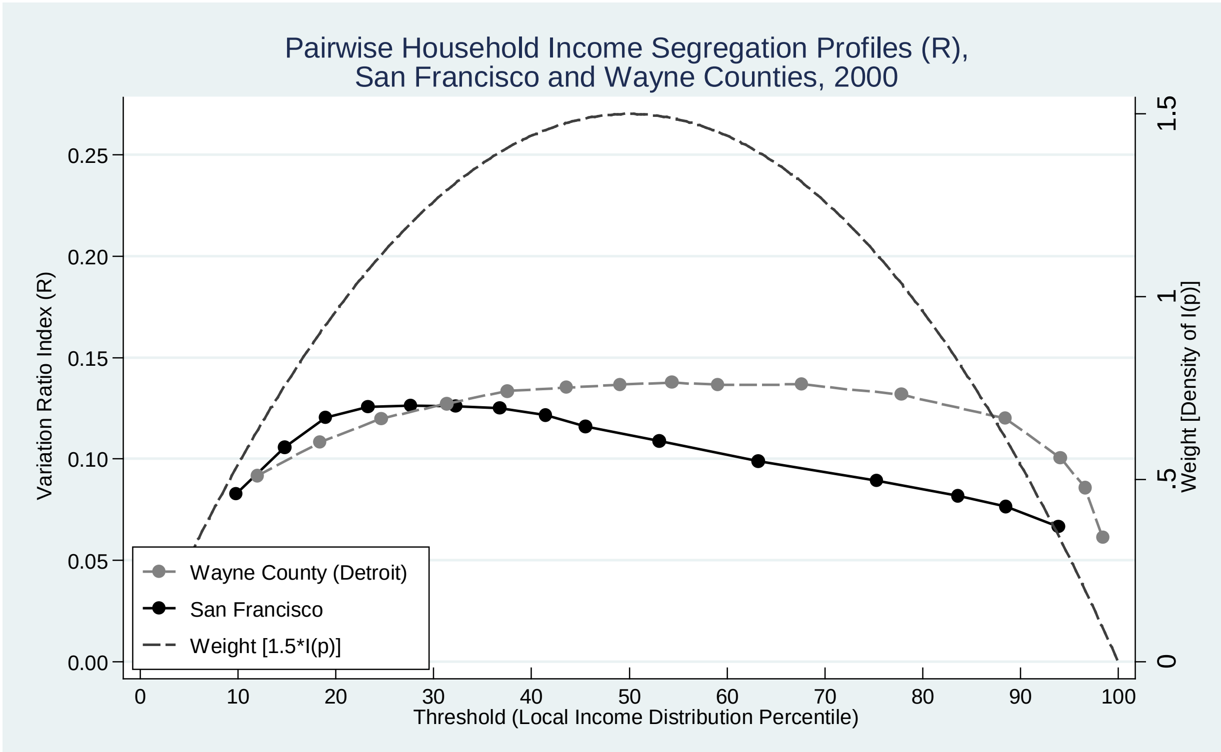

Pairwise Household Income Segregation Profiles (H),
San Francisco and Wayne Counties, 2000
Information Theory Index (H)
0.25
0.20
0.15
0.10
0.05
0.00
Weight [Density of E(p)]
1.5
1
.5
0
Wayne County (Detroit)
San Francisco
Weight [2*E(p)*ln(2)]
0
10
20
30
40
50
60
70
80
90
100
Threshold (Local Income Distribution Percentile)

**Figure 4**



Pairwise Household Income Segregation Profiles (R),
San Francisco and Wayne Counties, 2000
Variation Ratio Index (R)
0.25
0.20
0.15
0.10
0.05
0.00
Weight [Density of I(p)]
1.5
1
.5
0
0
10
20
30
40
50
60
70
80
90
100
Threshold (Local Income Distribution Percentile)
Wayne County (Detroit)
San Francisco
Weight [1.5*I(p)]

**Figure 5**

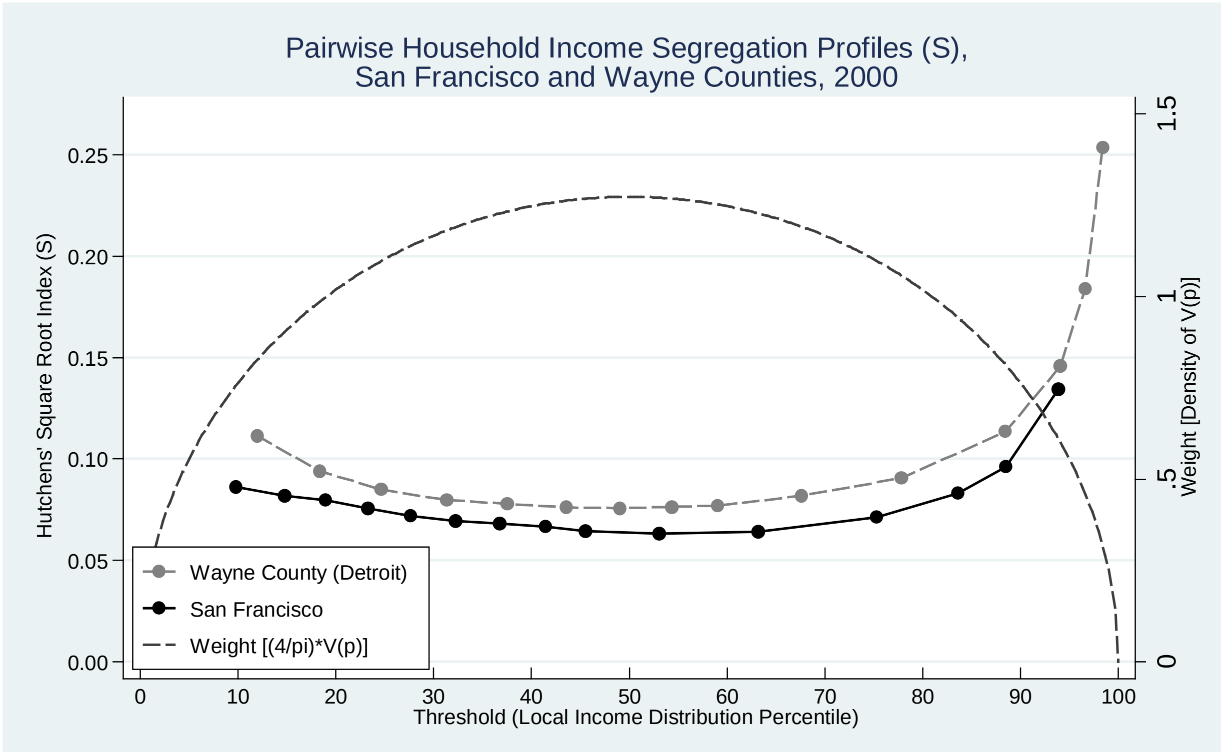
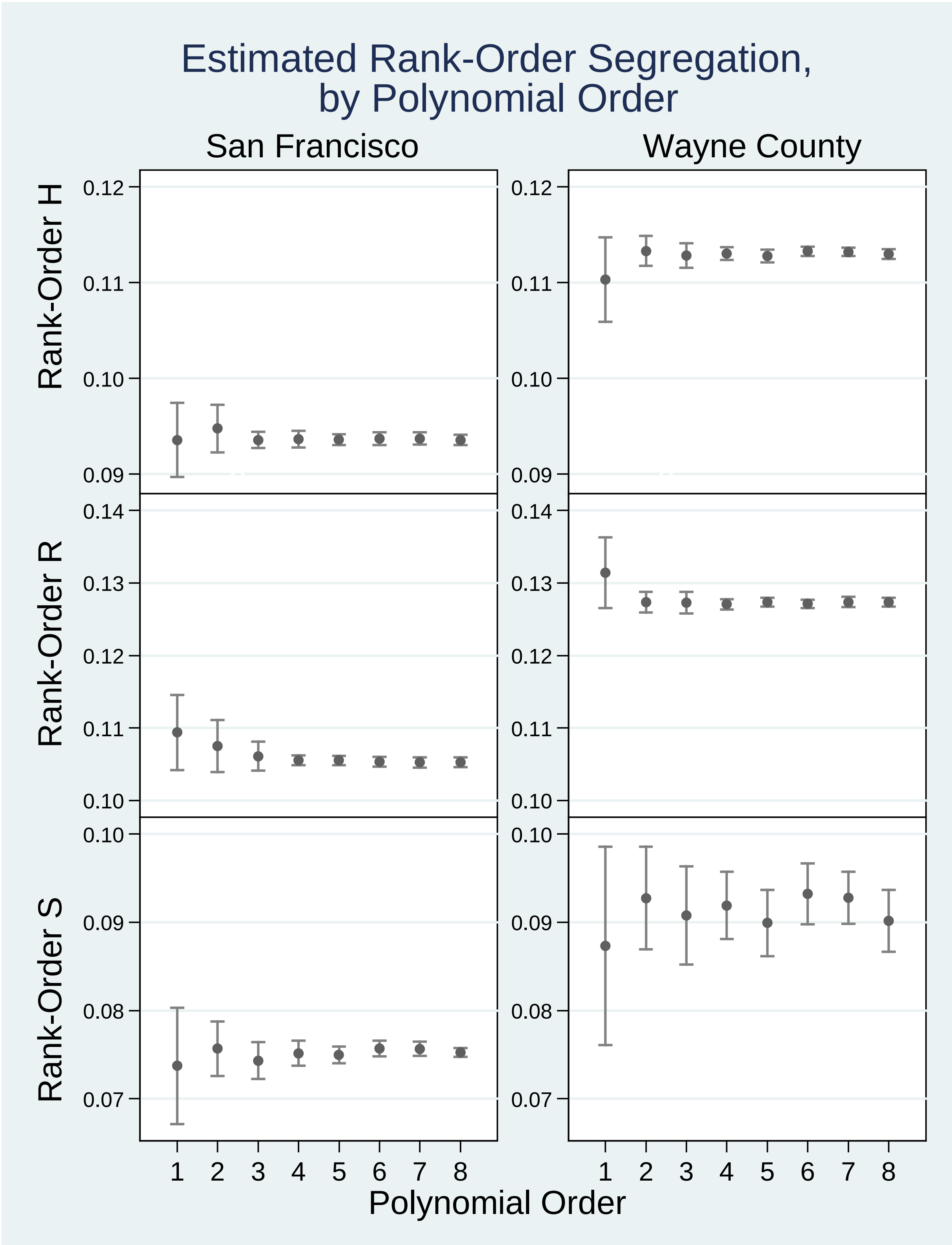
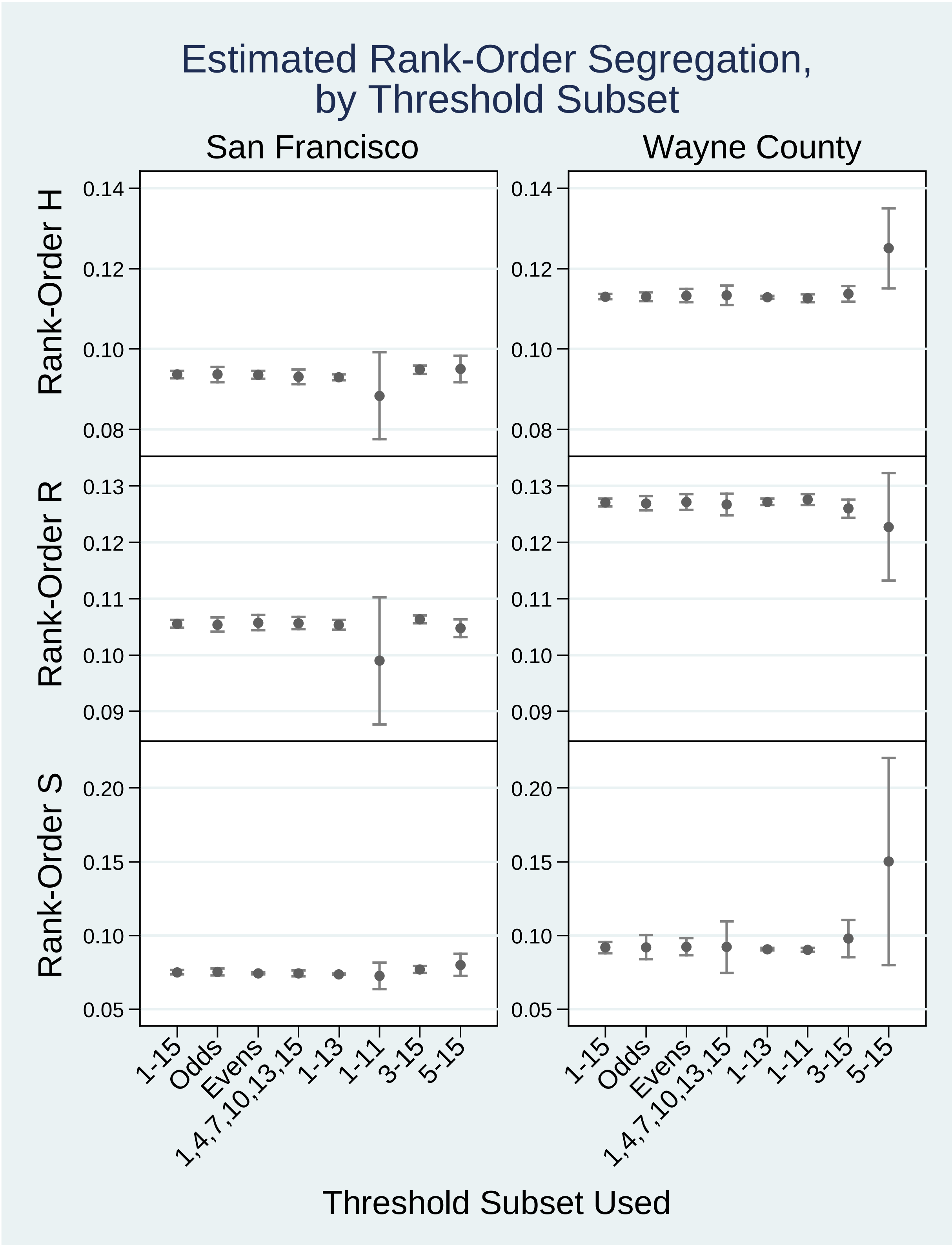

Pairwise Household Income Segregation Profiles (S),
San Francisco and Wayne Counties, 2000
Hutchens' Square Root Index (S)
0.25
0.20
0.15
0.10
0.05
0.00
Weight [Density of V(p)]
1.5
1
.5
0
Wayne County (Detroit)
San Francisco
Weight [(4/pi)*V(p)]
0
10
20
30
40
50
60
70
80
90
100
Threshold (Local Income Distribution Percentile)

**Figure 6**



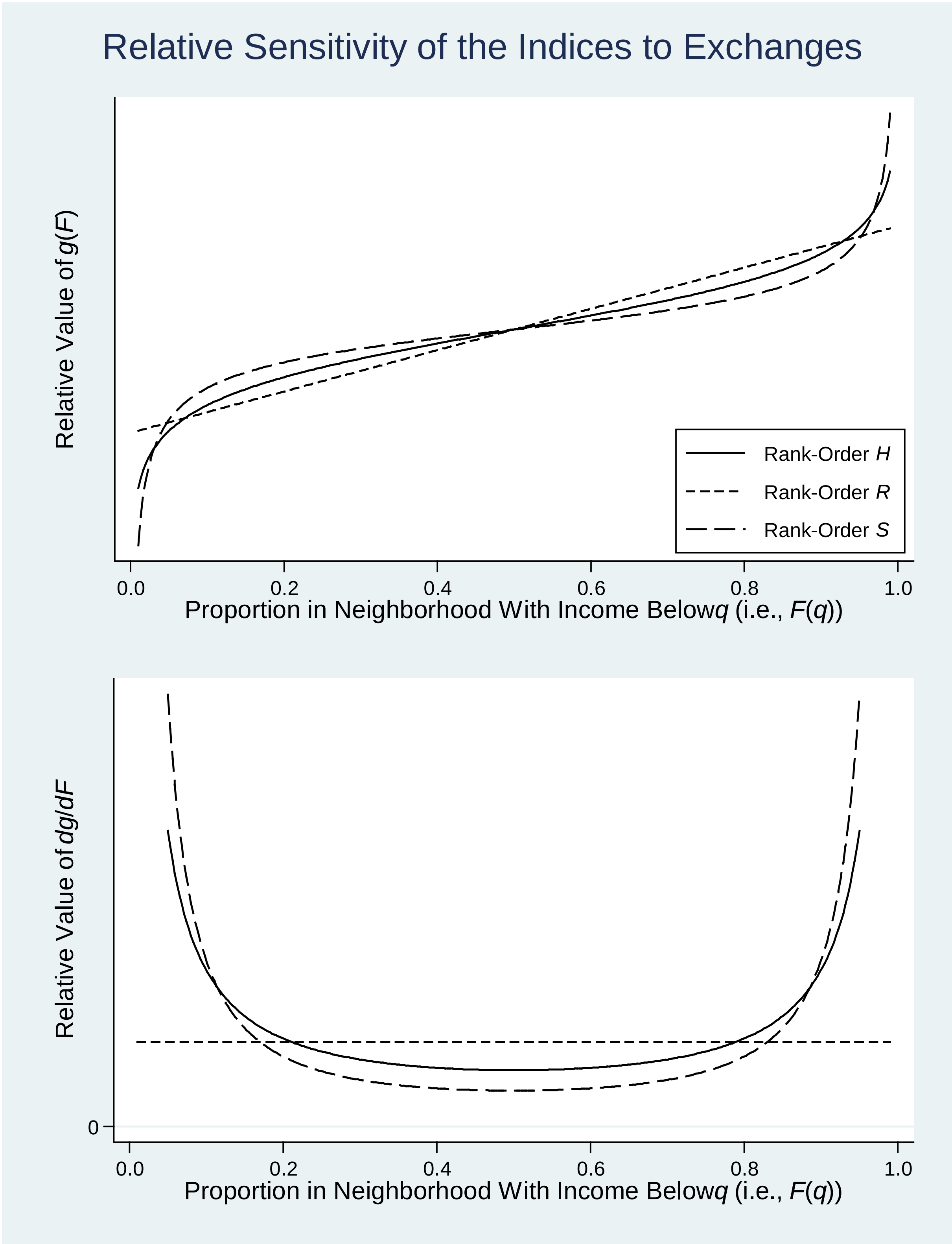
Estimated Rank-Order Segregation,
by Polynomial Order
San Francisco
Wayne County
Rank-Order H
Rank-Order R
Rank-Order S
0.12
0.11
0.10
0.09
0.14
0.13
0.12
0.11
0.10
0.10
0.09
0.08
0.07
1 2 3 4 5 6 7 8
Polynomial Order

**Figure 7**



Estimated Rank-Order Segregation,
by Threshold Subset
San Francisco
Wayne County
Rank-Order H
Rank-Order R
Rank-Order S
0.14
0.12
0.10
0.08
0.13
0.12
0.11
0.10
0.09
0.20
0.15
0.10
0.05
1-15
Odds
Evens
1,4,7,10,13,15
1-13
1-11
3-15
5-15
Threshold Subset Used

**Figure 8**



Relative Sensitivity of the Indices to Exchanges
Relative Value of g(F)
Rank-Order H
Rank-Order R
Rank-Order S
0.0
0.2
0.4
0.6
0.8
1.0
Proportion in Neighborhood With Income Belowq (i.e., F(q))
Relative Value of dg/dF
0
0.0
0.2
0.4
0.6
0.8
1.0
Proportion in Neighborhood With Income Belowq (i.e., F(q))

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